Coalgebraic Announcement Logics

Facundo Carreiro[1], Daniel Gorín[2], and Lutz Schröder[2]

[1] Institute for Logic, Language and Computation, Universiteit van Amsterdam
[2] Department of Computer Science, Universität Erlangen-Nürnberg

Abstract. In epistemic logic, dynamic operators describe the evolution of the knowledge of participating agents through communication, one of the most basic forms of communication being public announcement. Semantically, dynamic operators correspond to transformations of the underlying model. While metatheoretic results on dynamic epistemic logic so far are largely limited to the setting of Kripke models, there is evident interest in extending its scope to non-relational modalities capturing, e.g., uncertainty or collaboration. We develop a generic framework for non-relational dynamic logic by adding dynamic operators to coalgebraic logic. We discuss a range of examples and establish basic results including bisimulation invariance, complexity, and a small model property.

1 Introduction

Dynamic epistemic logics [5] are tools for reasoning about knowledge and belief of agents in a setting where interaction is of crucial interest. These logics extend epistemic logic (EL) [11] with dynamic operators, used to denote knowledge-changing actions. The most common of these is *public announcement*, first introduced in [17], which supports formulas of the form $\langle\phi\rangle\psi$ stating that after publicly (and faithfully) announcing that a certain fact ϕ holds (such as 'agent b does not know that agent a knows p'), ψ will hold (e.g. 'agent b knows p').

EL and its extension with public announcements (PAL) are typically interpreted on epistemic models, i.e. Kripke models where each accessibility relation is an equivalence; the points of the model represent *epistemic alternatives*. Evaluating a formula $\langle\phi\rangle\psi$ at a point c of an epistemic model I (notation $c \Vdash_I \langle\phi\rangle\psi$) amounts to verifying that the announcement is faithful (i.e., $c \Vdash_I \phi$) and that ψ holds at c after *removing* from I all epistemic alternatives where ϕ does *not* hold (notation $c \Vdash_{I\restriction\phi} \psi$). The term 'dynamic' refers precisely to the fact that models are changed during evaluation in this way.

Dynamic operators are of independent interest outside an epistemic setting. E.g., they occur as soon as one tries to express resiliency-related properties in verification (cf. van Benthem's *sabotage logic* [3] for an example); and they can turn a logic-based database query language into one supporting *hypothetical queries* (as in "return the aggregated sales we would have if we assumed that December sales corresponded to March").

Moreover, dynamic effects need not be restricted to a relational setting as found in Kripke models. E.g., the notion of announcing that a formula ψ holds has a natural analogue in probabilistic modal logic [12, 7, 10] where announcements have the effect of conditioning the current distribution, as discussed along with other examples of dynamic actions in non-relational settings by Baltag [1].

Of course, dynamic operators are subject to the usual tension between expressive power and computational complexity. The extension of robustly decidable modal languages with dynamic operators can quickly lead to undecidability (see e.g. [13, 18, 9]; also, the $\downarrow$-binder of hybrid logic can be seen as an example of a dynamic operator, and in general leads to undecidability). On the other hand, PAL is well-behaved: it is as expressive as EL but exponentially more succinct with the same complexity (PSPACE-complete in the multi-agent case [14, 8]).

In this paper we study announcement operators in a broad sense for modal logics beyond Kripke semantics. To deal with these at the appropriate level of generality, we work in the setting of coalgebraic modal logic [15], which uniformly covers a broad range of modal operators including, e.g., probabilistic, graded, and game-theoretic modalities (Sect. 2). A coalgebraic announcement can then be seen as the *global application* of a certain form of *local transformation* (contrasting with the global transformations considered in [1]).

A pervasive principle that transpires is that adding announcement operators preserves invariance under bisimulation and hence does not add fundamentally new expressive power; it may however be necessary to add new static modalities in order to eliminate announcements. We deal with generic announcement operators at increasing levels of generality, starting with a very well-behaved class of *strong announcements* that allow for a straightforward translation into the modal base language without requiring additional modalities. These constitute a particular type of *deterministic update* on models (i.e. certain transformation of the behaviour functor); which are also extended to account for announcements that are enriched with effects, such as non-determinism or uncertainty. As a unifying notion, we arrive at *backwards transformations* (Sect. 4) which act on predicates on the behaviour functor rather than the behaviour functor itself. We refer to the overall framework as *coalgebraic announcement logic* (CAL).

Besides bisimulation invariance, our technical results on these logics include: i) an equivalent translation of CAL into the modal base language, usually inducing an exponential blowup; ii) satisfiability preserving polynomial reductions to the base language for strong announcements, thus enabling the transfer of upper complexity bounds; and iii) a *constructive filtration* argument that yields a small model property independently of the presence of a master modality. The latter contribution appears to be a novel observation even for the static case, and substantially clarifies the original coalgebraic filtration construction [19].

2 Preliminaries

The framework of coalgebraic modal logics uniformly deals with a broad range of modal operators and a variety of different structures. This is achieved by recognizing the latter as instances of the concept of *coalgebra*. Given a functor $T : \mathsf{Set} \to \mathsf{Set}$, a *$T$-coalgebra* is a pair $\langle X, \gamma \rangle$ consisting of a non-empty set of *states* X and a *transition map* $\gamma : X \to TX$. We often identify $\langle X, \gamma\rangle$ with γ. For $x \in X$, we shall refer to $\gamma(x)$ as the *T-description of x*.

Example 1. Many structures that are well-known from theoretical computer science or from modal logic admit a natural presentation as coalgebras.

(i) Kripke frames are coalgebras for the covariant powerset functor $\mathcal{P}$. The map $\gamma_R : X \to \mathcal{P}X$ encodes a Kripke frame $\langle W, R\rangle$ with $\gamma_R(x) := \{w \mid xRw\}$.
(ii) A Kripke model $\langle W, R, V : \mathsf{P} \to \mathcal{P}W\rangle$ for a set P of propositions, corresponds to the K-coalgebra $\langle W, \gamma\rangle$ where $\mathsf{K}W := \mathcal{P}W \times \mathcal{P}\mathsf{P}$. The structure is recovered with $\gamma(x) := (\gamma_R(x), V^\flat(x))$ where $V^\flat(x) = \{p \in \mathsf{P} \mid x \in V(p)\}$.
(iii) The neighbourhood functor is $\mathcal{N} := \breve{\mathcal{P}} \circ \breve{\mathcal{P}}$, where $\breve{\mathcal{P}}$ is the contravariant powerset functor.[3] $\mathcal{N}$-coalgebras are the neighbourhood frames of modal logic [22], used in dynamic logics for reasoning about evidence and belief [6].
(iv) Let $\mathcal{M}$ be the subfunctor of $\mathcal{N}$ given by $\mathcal{M}X := \{S \in \mathcal{N}X \mid S \text{ is upwards closed}\}$. $\mathcal{M}$-coalgebras are monotone neighbourhood frames.
(v) The discrete distribution functor $\mathcal{D}_\omega$ maps X to the set of discrete probability distributions over X. $\mathcal{D}_\omega$-coalgebras are Markov chains. The subprobability functor $\mathcal{S}_\omega$ is similar but requires only that the measure of the whole space is at most 1 (instead of equal to 1).
(vi) Similarly, the finite multiset functor $\mathcal{B}_\omega$ maps X to the set of functions $\mu : X \to \mathbb{N}$ with finite support. Coalgebras for $\mathcal{B}_\omega$ are *multigraphs*, i.e. $\mathbb{N}$-weighted transition systems.

Coalgebras for a functor T form a category CoAlg_T where morphisms $f : \gamma \to \sigma$ between $\gamma : X \to TX$ and $\sigma : Y \to TY$ are maps $f : X \to Y$ with $\sigma \circ f = Tf \circ \gamma$. For $x \in X$ and $y \in Y$, we write $(x, \gamma) \sim (y, \sigma)$, read "$x$ and y are *behaviourally equivalent*", if there exists a coalgebra $\xi : Z \to TZ$ with morphisms $f : \gamma \to \xi$ and $g : \sigma \to \xi$ such that $f(x) = g(y)$. Functors are assumed wlog. to preserve injective maps [2] and to be non-trivial, in the sense that $TX = \emptyset$ implies $X = \emptyset$.

The syntax of coalgebraic modal logics is parametrized by a modal *similarity type* Λ. The language $\mathrm{CML}(\Lambda)$ is then given by the grammar

$$\phi ::= \bot \mid \phi \to \phi \mid \heartsuit_k(\phi_1, \dots \phi_k) \tag{1}$$

where $\heartsuit_k \in \Lambda$ is a modal operator of arity $k \geq 0$. We shall use the usual Boolean abbreviations $\wedge$, $\vee$, etc. when convenient. Each modality $\heartsuit_k \in \Lambda$ is interpreted as a k-ary *predicate lifting* $[\![\heartsuit_k]\!]$, i.e., a natural transformation $[\![\heartsuit_k]\!] : \breve{\mathcal{P}}^k \dot{\to} \breve{\mathcal{P}} \circ T^{op}$. The extension of ϕ in a coalgebra γ is given by $[\![\bot]\!]_\gamma = \emptyset$, $[\![\phi \to \psi]\!]_\gamma = (X \setminus [\![\phi]\!]_\gamma) \cup [\![\psi]\!]_\gamma$ and $[\![\heartsuit_k(\phi_1, \dots \phi_k)]\!]_\gamma = \{x \mid \gamma(x) \in [\![\heartsuit_k]\!]_X([\![\phi_1]\!]_\gamma, \dots, [\![\phi_k]\!]_\gamma)\}$. For the sake of readability, we sometimes pretend that all modal operators are unary.

Example 2. Some predicate liftings for the functors of Example 1 are

(i) For $\mathcal{P}$ we get the usual diamond with $[\![\Diamond]\!]_X(A) := \{t \in \mathcal{P}X \mid A \cap t \neq \emptyset\}$. The box is defined as $[\![\Box]\!]_X(A) := \{t \in \mathcal{P}X \mid t \subseteq A\}$.

[3] Formally, $\breve{\mathcal{P}} : \mathsf{Set}^{op} \to \mathsf{Set}$ with $\breve{\mathcal{P}}X = 2^X$ and, for $f : X \to Y$, $\breve{\mathcal{P}}f(A) = f^{-1}[A]$.

(ii) The diamond and box for K are obtained analogously. Propositions correspond to *nullary* liftings $[\![p]\!]_X := \{(s, C) \in \mathsf{K}X \mid p \in C\}$ for every $p \in \mathsf{P}$.
(iii) For $\mathcal{N}$ and $\mathcal{M}$, we have $[\![\Box]\!]_X(A) := \{s \in \mathcal{N}X \mid A \in s\}$.
(iv) For $\mathcal{D}_\omega$ (and $\mathcal{S}_\omega$), the modalities L_p of probabilistic modal logic correspond to the liftings $[\![L_p]\!]_X(A) := \{\mu \in \mathcal{D}_\omega X \mid \mu(A) \geq p\}$ for $p \in \mathbb{Q} \cap [0, 1]$.
(v) The counting modalities $\Diamond_k$ of graded modal logic are given as predicate liftings for $\mathcal{B}_\omega$ with $[\![\Diamond_k]\!]_X(A) := \{\mu \in \mathcal{B}_\omega X \mid \mu(A) \geq k\}$ for $k \in \mathbb{N}$.

It is well-known that $\mathrm{CML}(\Lambda)$ is invariant under behavioural equivalence; i.e., if $(x, \gamma) \sim (y, \sigma)$ then $x \in [\![\phi]\!]_\gamma$ iff $y \in [\![\phi]\!]_\sigma$, for all $\phi \in \mathrm{CML}(\Lambda)$.

An operator $\heartsuit \in \Lambda$ is *monotone* if $A \subseteq B \subseteq X$ implies $[\![\heartsuit]\!]_X A \subseteq [\![\heartsuit]\!]_X B$. For example, all operators of Example 2 are monotone except the one for $\mathcal{N}$. We say that Λ is *separating* [16] if $t \in TX$ is uniquely determined by $\{([\![\heartsuit]\!], A) \in \Lambda \times \breve{\mathcal{P}}X \mid t \in [\![\heartsuit]\!]_X(A)\}$.

3 Strong Coalgebraic Announcements

Announcements (and, more generally, dynamic operators) are accounted for at the syntactic level by extending $\mathrm{CML}(\Lambda)$ with a set Π of *dynamic modalities* which we call the *dynamic similarity type* (as opposed to *static modalities* of the *static similarity type* Λ). The syntax of $\mathrm{CAL}(\Pi, \Lambda)$ is obtained by extending the grammar in (1) with the clause $\Delta_{\phi_1}\phi_2$ for $\Delta \in \Pi$.

At this point, one may informally read $\Delta_\psi\phi$ as "after announcing ψ, ϕ holds"; more generally, Δ_ψ will represent an *update* operation on the model that is parameterized by a formula ψ. The update affects every state of the model but does so in a *local* way. That is, for each state x of γ, Δ_ψ updates $\gamma(x)$ in a way that depends only on $\gamma(x)$ and $[\![\psi]\!]_\gamma$.

Definition 3. An *update* is a natural transformation $\tau : T \dot{\to} (\breve{\mathcal{P}} \multimap T)$, where $\breve{\mathcal{P}} \multimap T$ is the Set-functor defined by $(\breve{\mathcal{P}} \multimap T)X := (TX)^{\breve{\mathcal{P}}X}$ and, for $f : X \to Y$, by $(\breve{\mathcal{P}} \multimap T)f := \lambda h : (TX)^{\breve{\mathcal{P}}X}.(Tf \circ h \circ \breve{\mathcal{P}}f)$.

Intuitively, every component τ_X takes as input the extension of a formula and the T-description of an element and returns an updated T-description. Naturality says that $Tf(\tau_X(t, \breve{\mathcal{P}}fA)) = \tau_Y(Tf(t), A)$, for $f : X \to Y$, $t \in TX$ and $A \subseteq Y$. We interpret $\Delta \in \Pi$ as an update $[\![\Delta]\!]$, and extend the semantics with the clause $[\![\Delta_\psi\phi]\!]_\gamma = [\![\phi]\!]_{[\![\Delta]\!]_X([\![\psi]\!]_\gamma)\circ\gamma}$ — i.e., Δ_ψ applies local changes to the entire coalgebra γ. We often identify Δ and $[\![\Delta]\!]$. The basic example is public announcement logic over unrestricted frames (as considered in [14]; which we call *standard PAL* although it is not interpreted over epistemic models), for which we take $[\![\Delta]\!](S)(A) = A \cap S$ and then rewrite an announcement $\langle\psi\rangle\phi$ to $\psi \wedge \Delta_\psi\phi$ — this induces essentially the standard semantics, since restricting all successors to satisfy ψ is modally indistinguishable from restricting the whole model to ψ.

Example 4. On relational models, the update $[\![!!]\!] : \mathcal{P} \dot{\to} (\breve{\mathcal{P}} \multimap \mathcal{P})$ defined as $[\![!!]\!]_X(S)(A) := \texttt{if } A \neq \emptyset \texttt{ then } S \cap A \texttt{ else } S$ gives the *total announcements* of [23]. That is, the announcement need not be truthful. If we think of A as the extension of a formula ϕ then this transformation removes the successors not satisfying ϕ. If an impossible formula is announced, it is ignored.

Example 5. For the functor $\mathcal{D}_\omega$ we can define an update $\tau : \mathcal{D}_\omega \dot{\to} (\breve{\mathcal{P}} \multimap \mathcal{D}_\omega)$ that has the effect of conditioning all probabilities to a given formula as $\tau_X(\mu)(A) := \lambda x.\mathtt{if}\ \mu(A) > 0\ \mathtt{then}\ \mu(x \mid A)\ \mathtt{else}\ \mu(x)$. Again, this update simply ignores the announcement of impossible events (i.e. those with probability 0). We also write this update as $\mu_{\restriction A} := \tau(\mu)(A)$.

It is clear that there is a resemblance between Examples 4 and 5: both updates give rise to dynamic operators that restrict the successors of a node to the points that satisfy certain formula. This connection can be made more precise, which will allow us to discuss this type of announcements in a uniform way.

Definition 6. An update τ is called a *strong announcement on Λ* if

(a) the partial application $\tau_X(-, A) : TX \to TX$ factors through the inclusion $i_A : TA \hookrightarrow TX$, for every $A \subseteq X$ (intuitively, $\tau_X(-, A) : TX \to TA$); and
(b) $\tau_X(s, A) \in [\![\heartsuit]\!]_X(C)$ iff $s \in [\![\heartsuit]\!]_X(C)$, for all $s \in TX$, $C \subseteq A \subseteq X$, $\heartsuit \in \Lambda$.

Condition (a) intuitively says that when ψ is announced, the resulting model should be based on the states satisfying ψ, while (b) ensures that all states satisfying ψ are retained. (Note that (b) is purely local and hence does not imply that whenever $\phi \to \psi$ is valid, then $\Delta_\psi \heartsuit\phi \leftrightarrow \heartsuit\phi$ is valid; this fails already in standard PAL.) In most cases, condition (b) is sufficient for naturality (so, for instance, we are exempt from proving it in the examples below).

Proposition 7. *A set-indexed family of maps $\tau_X : TX \to (2^X \to TX)$ satisfying condition (b) of Definition 6 for a separating set Λ of predicate liftings is a natural transformation $T \dot{\to} (\breve{\mathcal{P}} \multimap T)$; i.e. it is an update.*

Example 8. (i) In slight modification of Example 4, putting $[\![!]\!](S)(A) := A \cap S$ defines a strong announcement on $\{\Diamond\}$ (but not on $\{\Box\}$); it induces standard PAL. For the differences between $[\![!]\!]$ and $[\![!!]\!]$ see [14, 23].
(ii) Putting $\tau(\mu)(A) := \lambda x.\ \mathtt{if}\ x \in A\ \mathtt{then}\ \mu(x)\ \mathtt{else}\ 0$ for $\mu \in \mathcal{B}_\omega X$ defines a strong announcement on $\Lambda = \{\Diamond_0, \Diamond_1, \dots\}$. The case for the subdistribution functor $\mathcal{S}_\omega$ is similar.
(iii) For the neighbourhood functor $\mathcal{N}$, putting $\tau(t)(A) := t \cap \mathcal{P}A$ defines a strong announcement on $\Lambda = \{\Box\}$. The same definition (sic!) works for the monotone neighbourhood functor $\mathcal{M}$ and $\Lambda = \{\Diamond\}$.
(iv) Probabilistic conditioning (cf. Example 5) is *not* a strong announcement.

These examples show that strong announcements occur in varying settings. For monotone logics, they are actually uniquely determined.

Theorem 9. *Let Λ consist of monotone operators. If τ is a strong announcement on Λ, then we have an adjunction $Ti_A \dashv \tau_X(-, A)$ where the ordering on TX is given by $s \le t \iff \forall \heartsuit \in \Lambda, A \subseteq X.\, s \in [\![\heartsuit]\!](A) \Rightarrow t \in [\![\heartsuit]\!](A)$. In particular, τ is uniquely determined.*

This applies to all the updates of Example 8 except the one for $\mathcal{N}$ (since the predicate lifting involved is not monotone). In PAL, the announcement operator can be removed by means of well-known *reduction laws* [5, 14], and hence does not add expressive power. This generalizes to strong announcements:

Theorem 10. *Let Δ be a strong announcement on Λ, and let $\heartsuit \in \Lambda$; then $\Delta_\psi \heartsuit \phi \equiv \heartsuit(\psi \wedge \Delta_\psi \phi)$.*

Remark 11. Theorem 10 can be used on duals of strong announcements, yielding, e.g., that in standard PAL, $!_\psi \Box \phi \equiv !_\psi \neg \Diamond \neg \phi \equiv \neg !_\psi \Diamond \neg \phi \equiv \Box(\psi \rightarrow !_\psi \phi)$.

Corollary 12. *Let Π be a set of strong announcements on Λ. For every $\phi \in \mathrm{CAL}(\Pi, \Lambda)$ there is $\phi^* \in \mathrm{CML}(\Lambda)$ such that $\phi \equiv \phi^*$. Hence, $\mathrm{CAL}(\Pi, \Lambda)$ is invariant under behavioural equivalence.*

In general, the translation induced by Theorem 10 (and commutation of announcements with Boolean operators) induces an exponential blowup. In Section 5 we will look at this in more detail, and show that one can obtain a satisfiability-preserving polynomial translation in many cases.

4 Imperfect Announcements and Other Effects

The updates introduced in the previous section are all of a *deterministic* nature, in the sense that points of a coalgebra are updated in a unique way. While this is a sensible condition for many applications, one can also think of updates where the outcome is, e.g. non-deterministic or governed by a probability distribution (what are usually called *effects* in the context of programming languages).

Let us consider non-determinism for a moment. It seems reasonable to extend the (deterministic) updates of Definition 3 to natural transformations of the form $T \dot{\rightarrow} (\breve{\mathcal{P}} \multimap \mathcal{P}T)$ which return a set of possible T-descriptions to choose from. The question is now how to interpret a formula such as $\Delta_\psi \phi$ in this setting. Notice that there are at least two sensible readings: we could declare $\Delta_\psi \phi$ true at x in γ if ϕ is true at x in *all* possible transformations of $\gamma(x)$ (demonic interpretation); or, alternatively, if ϕ is true at x in *at least one* of them (angelic interpretation). This example shows that such a notion of non-deterministic update by itself does not suffice; we will see that the missing behaviour can be specified by means of predicate liftings.

In order to suitably generalize the deterministic updates of the previous section we involve a different notion of transformation that acts directly on the involved predicates. As such, unsurprisingly, it is contravariant, generalizing as it does the preimage under an update.

Definition 13. A *regenerator* is a natural transformation $\rho : \breve{\mathcal{P}} \times \breve{\mathcal{P}}T \dot{\rightarrow} \breve{\mathcal{P}}T$.

The arguments of a regenerator should be thought of as the extension of the announced formula ψ and a predicate on TX of the form $[\![\heartsuit]\!][\![\phi]\!]_{\gamma\restriction\psi}$, where $\gamma\restriction\psi$ denotes an updated version of γ; the regenerator transforms this back into a predicate on TX as seen from the original γ. We can now define the *coalgebraic logic of announcements with effects* $\mathrm{CAL}^\circ(\Pi, \Lambda)$, which syntactically coincides with $\mathrm{CAL}(\Pi, \Lambda)$. In $\mathrm{CAL}^\circ(\Pi, \Lambda)$, each $\Delta \in \Pi$ is interpreted by a regenerator $[\![\Delta]\!]^\circ$. The semantics of formulas requires not only a T-coalgebra $\langle X, \gamma \rangle$ but also a map $\rho : 2^{TX} \to 2^{TX}$ (the *global regenerator*) that keeps track of the updates applied so far. The extension $[\![\cdot]\!]^\circ_{\rho,\gamma}$ of formulas of $\mathrm{CAL}^\circ(\Pi, \Lambda)$ is defined as usual for Boolean connectives and by

$$[\![\Delta_\psi \phi]\!]^\circ_{\rho,\gamma} = [\![\phi]\!]^\circ_{[\![\Delta]\!]^\circ_X([\![\psi]\!]^\circ_{\rho,\gamma}, -)\circ\rho,\gamma} \qquad [\![\heartsuit \phi]\!]^\circ_{\rho,\gamma} = (\breve{\mathcal{P}}\gamma \circ \rho \circ [\![\heartsuit]\!]_X)[\![\phi]\!]^\circ_{\rho,\gamma} \ .$$

When no ambiguity arises, we may write $[\![\cdot]\!]$ instead of $[\![\cdot]\!]^\circ$. We will also use $[\![\phi]\!]_\gamma$ instead of $[\![\phi]\!]_{\iota_X,\gamma}$ where $\iota : \breve{\mathcal{P}}TX \dot{\to} \breve{\mathcal{P}}T$ is the identity.

The connection between regenerators and "updates with effects" as in the non-deterministic update discussed above can now be made precise. The crucial observation is that any natural transformation $\tau : T \dot{\to} (\breve{\mathcal{P}} \twoheadrightarrow FT)$ equipped with a predicate lifting (for F) $\lambda : \breve{\mathcal{P}} \dot{\to} \breve{\mathcal{P}}F$ induces the regenerator (for T) $\rho^{\tau,\lambda} : \breve{\mathcal{P}} \times \breve{\mathcal{P}}T \dot{\to} \breve{\mathcal{P}}T$ defined by $\rho^{\tau,\lambda}_X(A, S) := \breve{\mathcal{P}}(\tau_X(-)(A))[\lambda_{TX}(S)]$. In fact, $\mathrm{CAL}(\Lambda, \Pi)$ is just $\mathrm{CAL}^\circ(\Lambda, \Pi)$ with $F = \mathrm{Id}$ and $\lambda = id$.

Example 14. The non-deterministic announcements discussed above correspond to taking $F = \mathcal{P}$; the angelic interpretation is induced by $\lambda^a_X(t) := \{s \mid t \cap s \neq \emptyset\}$ and the demonic one by $\lambda^d_X(t) := \{s \mid t \subseteq s\}$ (i.e. $[\![\Diamond]\!]$ and $[\![\Box]\!]$ from Example 2.i). Examples of other updates for various choices of T and τ are:

(i) *Lossy announcements*: take $T = \mathcal{P}$ and $\tau_X(S, A) := \{S \cap A, S\}$; this models an announcement that can fail (leaving the set of successors unchanged). If $[\![\Delta]\!]^\circ$ is based on λ^d, then $\Delta_\psi \phi$ means that ϕ has to hold regardless of whether the announcement of ψ succeeds or not. The angelic case is dual.

(ii) *Controlled sabotage*: again for $T = \mathcal{P}$, but define $\tau_X(S, A) := \{S \setminus A, S\}$. If we think of A as a delicate area of a network, this transformation models links that may fail every time we want to go through them.

(iii) *Unstable (pseudo-)Markov chains*: let $T = \mathcal{S}_\omega$ and, for each $\varepsilon \in \mathbb{Q} \cap [0, 1]$, define a non-deterministic update $\tau^\varepsilon_X(\mu, A) = \{\tilde{\mu}_p \mid 0 \leq p \leq \varepsilon, \tilde{\mu}_p \in \mathcal{S}_\omega X\}$ where $\tilde{\mu}_p(x) := \texttt{if } x \in A \texttt{ then } \mu(x) + p \texttt{ else } \mu(x)$. This update non-deterministically augments the probability of each $a \in A$ by at most ε.

Example 15. Taking $F = \mathcal{D}_\omega$ we get a probability distribution over the outcomes of an update. For $p \in \mathbb{Q} \cap [0, 1]$ we can define $\lambda^p_X(A) := \{\mu \mid \mu(A) \geq p\}$, obtaining dynamic operators Δ^p such that $\Delta^p_\psi \phi$ is true if the probability of the effect of announcing ψ (in some unspecified way) making ϕ true is greater than p. Note that the underlying coalgebra need not be probabilistic: in this example the coalgebra type T is arbitrary and F only plays a role in the liftings.

Remark 16. One is tempted to think of non-deterministic or probabilistic updates as changing the coalgebra γ, non-deterministically or randomly, to a fixed γ'. Although this is not accurate in that the choice is made again *every time* the evaluation encounters a static modality, it becomes formally correct by restricting to *tree-shaped* coalgebras, i.e. those where the underlying Kripke frame is a tree, which, in the light of Theorem 17 below, is without loss of generality since every coalgebra is behaviourally equivalent to a tree-shaped one [20]. One still needs to keep in mind, however, that the choice is made *per state*, e.g. a lossy announcement may succeed in some states and fail in others.

We now show that even in the presence of effects, dynamic modalities can be rewritten in terms of static modalities (albeit not necessarily of the base logic), and hence coalgebraic announcement logic in the more general sense remains invariant under behavioural equivalence. The crucial observation is that composing a predicate lifting and a regenerator yields a predicate lifting of a higher arity. That is, given $\lambda : \breve{\mathcal{P}}^n \dot{\to} \breve{\mathcal{P}}T$ and $\rho : \breve{\mathcal{P}} \times \breve{\mathcal{P}}T \dot{\to} \breve{\mathcal{P}}T$, we have that the composite $\lambda'_X(A, B_1 \dots B_n) := \rho_X(A, \lambda_X(B_1 \dots B_n))$ is a predicate lifting $\lambda' : \breve{\mathcal{P}}^{n+1} \dot{\to} \breve{\mathcal{P}}T$. Given a static modality $\heartsuit$ and a dynamic modality Δ, we introduce a static modality $\boxtimes_{(\Delta\cdot\heartsuit)}$ interpreted by the composite of $[\![\Delta]\!]$ and $[\![\heartsuit]\!]$ in this sense; one easily shows that

$$\Delta_\psi \heartsuit(\phi_1, \dots, \phi_n) \equiv \boxtimes_{(\Delta\cdot\heartsuit)}(\psi, \Delta_\psi\phi_1, \dots, \Delta_\psi\phi_n).$$

Iterating this, we obtain static modalities $\boxtimes_m$ for all strings $m = \Delta^1 \cdots \Delta^n \heartsuit$; we denote the similarity type extending Λ by these modalities as $\mathsf{CL}_\Pi(\Lambda)$. We say that Λ is *closed for* Π if for every $\boxtimes_m \in \mathsf{CL}_\Pi(\Lambda)$, $\boxtimes_m(a_1, \dots, a_n)$ (for propositional variables a_i) can be expressed as a polynomial-sized (in n) formula that is a propositional combination of formulae $\heartsuit\phi$ where ϕ is a propositional combination of the a_i. Note that when Π consists of strong announcements on Λ, then Λ is closed for Π.

Theorem 17. *For all* $\phi \in \mathrm{CAL}^\circ(\Pi, \Lambda)$ *there is* $\phi^* \in \mathrm{CML}(\mathsf{CL}_\Pi(\Lambda))$ *s.t.* $\phi \equiv \phi^*$. *Hence,* $\mathrm{CAL}^\circ(\Pi, \Lambda)$ *is invariant under behavioural equivalence.*

5 Decidability and Complexity

We have shown in the previous sections that coalgebraic announcement logic can be reduced to basic coalgebraic modal logic, albeit incurring an exponential blow-up. For PAL, it is known that this blow-up is unavoidable [14, 8] and yet its computational complexity is the same as that of the base logic. We now show that under mild assumptions, the complexity result generalizes to the coalgebraic setting. We consider two standard decision problems: i) the *satisfiability problem* (SAT) "given a formula ϕ, decide if there is $\langle X, \gamma\rangle$ such that $[\![\phi]\!]_\gamma \neq \emptyset$"; and ii) the *constrained satisfiability problem* (CSAT) "given two formulas ψ and ϕ, decide if there is $\langle X, \gamma\rangle$ such that $[\![\phi]\!]_\gamma \neq \emptyset$ and $[\![\psi]\!]_\gamma = X$", in which case we say that ϕ is *satisfiable with respect to* ψ. In the terminology of description logic, CSAT corresponds to reasoning with a general TBox (given by ψ). SAT is a special case of CSAT but tends to have lower complexity.

Our results have different levels of generality. First we prove a small model property which holds unconditionally. For the case that the static similarity type Λ is closed for Π, we moreover provide a polynomial reduction of CSAT to the base logic, which allows inheriting the complexity of CSAT for the latter, typically EXPTIME. For a polynomial reduction of SAT to the base logic, we need to assume that Λ contains a *master modality*, which then again allows inheriting the complexity, typically PSPACE. We illustrate these methods for the logic of probabilistic conditioning.

5.1 Constructive Filtrations and the Small Model Property

For Σ a set of formulas, a Σ-filtered model is understood as one whose states are subsets of Σ and such that each state satisfies all the Σ-formulas it contains. Hence, a Σ-filtered model has at most $2^{|\Sigma|}$ states. We shall prove that every satisfiable formula ϕ of $\mathrm{CAL}^\circ(\Lambda, \Pi)$ is satisfied in some Σ-filtered model with $|\Sigma| = O(|\phi|)$. In fact, we show how to derive a Σ-filtered models from any model for ϕ, rather simplifying the construction given for $\mathrm{CML}(\Lambda)$ in [19].

In what follows, let Σ be a fixed set of $\mathrm{CAL}^\circ(\Lambda, \Pi)$-formulas, closed under subformulas and negation (identifying $\neg\neg\phi$ with ϕ as usual). Let $H_\Sigma \subseteq 2^\Sigma$ be the set of all maximal satisfiable subsets of Σ. For a given coalgebra $\gamma : X \to TX$, let $f_\Sigma : X \to H_\Sigma$ be the mapping $f_\Sigma(x) = \{\phi \in \Sigma \mid x \in [\![\phi]\!]_\gamma\}$. A coalgebra $\gamma_\Sigma : H_\Sigma \to TH_\Sigma$ is said to be a *Σ-filtration* of γ whenever for all $x \in X$, there exists $y \in X$ such that $f_\Sigma(x) = f_\Sigma(y)$ and $\gamma_\Sigma(f_\Sigma(x)) = Tf_\Sigma(\gamma(y))$.

Intuitively, we take the quotient of X by satisfaction of formulas in Σ and allow *any* of the members of the equivalence class to be the representative (each choice of representatives induces a potentially different filtration).[4]

To state the filtration theorem we need to relate global regenerators based on a coalgebra γ with global regenerators based on a Σ-filtration γ_Σ. So we say that two maps $\rho_X : 2^{TX} \to 2^{TX}$ and $\rho_{H_\Sigma} : 2^{TH_\Sigma} \to 2^{TH_\Sigma}$ are *f_Σ-synchronized* if $\rho_X \circ \breve{\mathcal{P}}Tf = \breve{\mathcal{P}}Tf \circ \rho_{H_\Sigma}$ (the naturality diagram for f_Σ if ρ was a natural transform). By induction over ϕ, one shows

Theorem 18. *Let γ_Σ be a Σ-filtration of $\langle X, \gamma\rangle$. For all $\phi \in \Sigma$, $x \in X$ and every pair of f_Σ-synchronized ρ_X and ρ_{H_Σ}, we have $x \in [\![\phi]\!]_{\rho_X,\gamma}$ iff $f_\Sigma(x) \in [\![\phi]\!]_{\rho_{H_\Sigma},\gamma_\Sigma}$.*

Observing that $id_{2^{TX}}$ and $id_{2^{TH_\Sigma}}$ are f_Σ-synchronized, we obtain

Corollary 19. $\mathrm{CAL}^\circ(\Lambda, \Pi)$ *has the small (exponential) model property.*

It is easy to exploit the small model property to give an upper bound NEXPTIME for CSAT under mild additional conditions; in view of the results in the next section, we refrain from spelling out details.

5.2 Polynomial Satisfiability-Preserving Translations

From Theorem 17 we already know that when Λ is closed for Π, then every formula in $\mathrm{CAL}^\circ(\Lambda, \Pi)$ is equivalent to one of $\mathrm{CML}(\Lambda)$, perhaps exponentially

[4] The simpler definition $\gamma_\Sigma(f_\Sigma(x)) = Tf_\Sigma(\gamma(x))$ is not well-defined when f_Σ is not injective. Also, f_Σ may not be a coalgebra morphism, even with $\Sigma = \mathrm{CAL}^\circ(\Lambda, \Pi)$.

larger. This implies that the complexity of the decision problems for $\mathrm{CAL}^\circ(\Lambda, \Pi)$ is at most one exponential higher than for $\mathrm{CML}(\Lambda)$. But one can do better. The main observation is that, although the translated formula ϕ^* may be of size exponential in $|\phi|$, it contains only polynomially many *different* subformulas. Using essentially the same argument as in [14, Lemma 9], one can prove:

Theorem 20. *Let Λ be closed for Π. Then* CSAT *for* $\mathrm{CAL}^\circ(\Lambda, \Pi)$ *has the same complexity as for* $\mathrm{CML}(\Lambda)$*.*

The proof is by introducing propositional variables as abbreviations for subformulas, using the constraint. To deal with satisfiability in the absence of a constraint, we need a master modality to make abbreviations work up to the modal depth of the target formula. Coalgebraically, a *master modality* for Λ is a static modality $\boxdot$ such that $\boxdot\top$ and $\boxdot\phi \to (\heartsuit\psi \leftrightarrow \heartsuit(\phi \wedge \psi))$, for all $\heartsuit \in \Lambda$ and $\phi, \psi \in \mathrm{CML}(\Lambda)$, are valid. In the presence of a master modality one can give better bounds for SAT than those from Theorem 20.

Theorem 21. *Let Λ be closed for Π, and contain a master modality. Then the complexity of* SAT *for* $\mathrm{CAL}^\circ(\Lambda, \Pi)$ *is the same as for* $\mathrm{CML}(\Lambda)$*.*

Interestingly, master modalities abound: if T preserves inverse images then the predicate lifting $[\![\boxdot]\!]_X(A) := TA$ induces a master modality. Preserving inverse images is weaker than the frequent assumption of preservation of weak pullbacks. E.g., in graded modal logic $\Box_1 := \neg\Diamond_1\neg$ is a master modality, and in probabilistic modal logic L_1 is a master modality. Having observed that Λ is closed for strong announcements on Λ, we note explicitly

Theorem 22. *If Π consists of strong announcements on Λ, then* CSAT *for* $\mathrm{CAL}(\Lambda, \Pi)$ *has the same complexity as for* $\mathrm{CML}(\Lambda)$*; the same holds for* SAT *if Λ contains a master modality.*

In particular, we regain the known complexity of standard PAL, and we obtain, as new results, PSPACE and EXPTIME as the complexity of SAT and CSAT, respectively, for graded modal logic with the strong announcement operator (Example 8), as well as, e.g., NP as the complexity of neighbourhood logic and monotone modal logic with strong announcement.

5.3 Case Study: Conditionings in Probabilistic Logic

We now turn the attention to a logic where a master modality is available but the announcements that we are interested in are not strong: the logic of probabilistic conditioning (cf. Examples 5 and 8), the latter denoted Δ, with L_1 being the master modality.

First, we observe that the static similarity type $\Lambda = \{L_p \mid p \in \mathbb{Q} \cap [0, 1]\}$ likely fails to be closed for $\{\Delta\}$. To see this, we first move to an extended modal language with linear inequalities over probabilities of formulas ϕ, the latter denoted

$\ell(\phi)$ for 'likelihood' (i.e. essentially the probabilistic part of the logic introduced in [7]). In this notation, we have

$$\Delta_\psi L_p\phi \equiv (\ell(\psi) = 0 \to \ell(\Delta_\psi\phi) \geq 0) \wedge \ell(\psi \wedge \Delta_\psi\phi) \geq p \cdot \ell(\psi) \tag{2}$$

where the first conjunct takes care of the exceptional case of impossible announcements. It seems unlikely that one could express the right-hand-side of (2) with a finite formula using only the operators L_p. However, we can extend Λ to a closed similarity type. A very conservative solution is to let $L_p(\phi \mid \psi)$ be a binary modal operator abbreviating $\ell(\phi \wedge \psi) \geq p \cdot \ell(\psi)$; then

$$\Delta_\psi L_p(\phi \mid \chi) \equiv (L_1(\neg\psi \mid \top) \to L_p(\Delta_\psi\phi \mid \chi)) \wedge L_p(\Delta_\psi\phi \mid \chi \wedge \psi),$$

i.e. the $L_p(-,-)$ are closed for $\{\Delta\}$. More generally, one may verify that the full language of linear inequalities (with n-ary modal operators $\sum_{i=1}^n a_i\ell(\_\_{}_i) \geq b$ for all $n \geq 0$ and $a_1, \ldots, a_n, b \in \mathbb{Q}$) is closed. SAT for the modal logic of linear inequalities over probabilities is known to be in PSPACE [7], hence the complexity of SAT for the above logics of probabilistic conditioning is PSPACE.

6 Conclusions

We have introduced the framework of coalgebraic announcement logics and seen that it transfers to a setting of richer structures and general effects many nice properties enjoyed by (relational) public announcement logics. Our work fits in the spirit of [1], which also studies dynamic epistemic operators on a coalgebraic setting; although with a rather different perspective and a completely different technical machinery. That framework gains much generality from defining updates as natural transformations in CoAlg_T instead of Set. Giving up locality in this way (for now updates can look at the whole coalgebra structure) has its consequences: one loses small (or even finite) model properties, general decidability results, etc. The framework of [4] avoids explicit updates of models but otherwise has a comparable level of generality, with similar advantages and drawbacks. We expect that all coalgebraic announcement logics can be shown to be expressible in those frameworks.

It is only a slight simplification to claim that *all coalgebraic results are compositional.* One can reduce the study of composite functors to that of multi-sorted functors and almost all coalgebraic results extend straightforwardly from the single-sorted to the multi-sorted one at the expense of no more than additional indexes in the notation [21]. Applying this mechanism to the case of coalgebraic announcement logics requires some concentration (e.g. one needs to realize that regenerators apply to multi-sorted predicates) but does not pose any essential problems. Effectively, this means that all our results — invariance under behavioural equivalence, complexity analysis and the small model property — carry over to (complex) composite settings, such as a probabilistic logic about beliefs in a multi-agent system with group announcement operators that communicate facts only to selected agents by probabilistic conditioning.

References

1. Baltag, A.: A coalgebraic semantics for epistemic programs. In: Coalgebraic Methods in Computer Science. ENTCS, vol. 82, pp. 17–38. Elsevier (2003)
2. Barr, M.: Terminal coalgebras in well-founded set theory. Theoret. Comput. Sci. 114, 299–315 (1993)
3. van Benthem, J.: An essay on sabotage and obstruction. In: Mechanizing mathematical reasoning. LNCS, vol. 2605, pp. 268–276. Springer (2005)
4. Cîrstea, C., Sadrzadeh, M.: Coalgebraic epistemic update without change of model. In: Alg. and Coalg. in Comp. Sci. LNCS, vol. 4624, pp. 158–172. Springer (2007)
5. van Ditmarsch, H., van der Hoek, W., Kooi, B.: Dynamic epistemic logics. Springer (2007)
6. Duque, D.F., van Benthem, J., Pacuit, E.: Evidence logic: a new look at neighborhood structures. In: Advances in Modal Logics. College Publications (2012)
7. Fagin, R., Halpern, J.Y.: Reasoning about knowledge and probability. J. ACM 41, 340–367 (1994)
8. French, T., van der Hoek, W., Iliev, P., Kooi, B.: Succinctness of epistemic languages. In: Int. Joint Conf. on Artif. Int. pp. 881–886 (2011)
9. French, T., van Ditmarsch, H.: Undecidability for arbitrary public announcement logic. In: Advances in Modal Logics. pp. 23–42. College Publications (2008)
10. Heifetz, A., Mongin, P.: Probabilistic logic for type spaces. Games and Economic Behavior 35, 31–53 (2001)
11. Hintikka, J.: Knowledge and belief. Cornell University Press (1962)
12. Larsen, K., Skou, A.: Bisimulation through probabilistic testing. Inf. Comput. 94, 1–28 (1991)
13. Löding, C., Rohde, P.: Model checking and satisfiability for sabotage modal logic. In: FSTTCS. LNCS, vol. 2914, pp. 302–313. Springer (2003)
14. Lutz, C.: Complexity and succinctness of public announcement logic. In: Joint Conference on Autonomous Agents and Multi-Agent Systems. pp. 137–143 (2006)
15. Pattinson, D.: Coalgebraic modal logic: Soundness, completeness and decidability of local consequence. Theoret. Comput. Sci. 309, 177–193 (2003)
16. Pattinson, D.: Expressive logics for coalgebras via terminal sequence induction. Notre Dame J. Formal Logic 45, 2004 (2002)
17. Plaza, J.A.: Logics of public communications. In: International Symposium on Methodologies for Intelligent Systems. pp. 201–216 (1989)
18. Rohde, P.: Moving in a crumbling network: The balanced case. In: Computer Science Logic. LNCS, vol. 3210, pp. 310–324. Springer (2004)
19. Schröder, L.: A finite model construction for coalgebraic modal logic. J. Log. Algebr. Prog. 73, 97–110 (2007)
20. Schröder, L., Pattinson, D.: Coalgebraic correspondence theory. In: FoSSaCS 2010. LNCS, vol. 6014, pp. 328–342. Springer (2010)
21. Schröder, L., Pattinson, D.: Modular algorithms for heterogeneous modal logics via multi-sorted coalgebra. Math. Struct. Comput. Sci. 21(2), 235–266 (2011)
22. Segerberg, K.: An essay in classical modal logic. No. 1 in Filosofiska studier utgivna av Filosofiska föreningen och Filosofiska institutionen vid Uppsala univ. (1971)
23. Steiner, D., Studer, T.: Total public announcements. In: Logical Foundations of Computer Science. pp. 498–511. Springer (2007)