Basic Model Theory for Memory Logics

Carlos Areces[1], Facundo Carreiro[⋆2], Santiago Figueira[⋆⋆3] and Sergio Mera[⋆2]

[1] INRIA Nancy Grand Est, Nancy, France,
areces@loria.fr,
[2] Departamento de Computación, FCEyN, UBA, Argentina,
[3] Departamento de Computación, FCEyN, UBA, and CONICET, Argentina
{fcarreiro, sfigueir, smera}@dc.uba.ar

Abstract. Memory logics is a family of modal logics whose semantics is specified in terms of relational models enriched with additional *data structure* to represent a memory. The logical language includes a collection of operations to access and modify the data structure. In this paper we study basic model properties of memory logics, and prove results concerning characterization, definability and interpolation. While the first two properties hold for all memory logics introduced in this article, interpolation fails in most cases.

1 Introduction

In the last decades, *modal logics* have become a wide collection of languages with some general aspects in common: they are usually interpreted over relational structures, they are generally computationally well behaved, and they take a local perspective when evaluating a formula. Nowadays, the practical influence of modal logics is undeniable as they are used in many applications like linguistics, artificial intelligence, knowledge representation, specification and software verification, etc. (see [1] for details).

In a number of recent papers [2,3,4,5] we have investigated a family of logics called *memory logics*, extending the classical modal logic[4]. Intuitively, memory logics enrich the standard relational models used by most modal logics with a *data structure*. The logical language is then extended with a collection of operations to access and modify this data structure. In this article we fix the data structure to be a *set*, but other structures are analyzed in [5].

Assume as given a signature $\mathcal{S} = \langle \textsc{prop}, \textsc{rel} \rangle$ that defines the sets of propositional and relational symbols, respectively. Let $\mathcal{N}$ be a standard relational model over $\mathcal{S}$, i.e., $\mathcal{N} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F} = \langle W, (R_r)_{r \in \textsc{rel}} \rangle$ is a suitable frame (i.e. W is a nonempty set whose elements we will call *states*, and $R_r \subseteq W^2$ for each $r \in \textsc{rel}$, which we call *accessibility relations*) and $V : \textsc{prop} \to 2^W$ is the valuation function. We obtain a model for memory logics, extending this structure

⋆ F. Carreiro and S. Mera were partially supported by CONICET Argentina.
⋆⋆ S. Figueira was partially supported by CONICET Argentina (grant PIP 370).
[4] Due to lack of space we will assume in this article that the reader is familiar with modal logics, see [1,6] for complete details.

with a set $S \subseteq W$ representing the current 'memory' of the model. For $\mathcal{M}$ an arbitrary model, we will denote its domain by $|\mathcal{M}|$, and we will usually represent a model $\langle\langle W, R\rangle, V, S\rangle$ simply as $\langle W, R, V, S\rangle$.

A set is a very simple data structure (e.g., compare it with a list, a tree, etc), but even in this setting, we can define a set of operators that interacts with the memory in different ways. One can think of different types of simple updates that can be performed on the memory of a model: to store or delete an element, to clean the memory, etc. If $\mathcal{M} = \langle\mathcal{F}, V, S\rangle$ is a model for memory logics as defined above, we define

$$\mathcal{M}[*] = \langle\mathcal{F}, V, \emptyset\rangle; \quad \mathcal{M}[w] = \langle\mathcal{F}, V, S \cup \{w\}\rangle; \quad \mathcal{M}[-w] = \langle\mathcal{F}, V, S \setminus \{w\}\rangle.$$

Let $\mathcal{M}[w_1, \ldots, w_n]$ be a shorthand for $((\mathcal{M}[w_1]) \ldots)[w_n]$. Besides the standard Boolean and diamond operators of the basic modal logic, we define

$$\begin{array}{llll} \mathcal{M}, w \models \textcircled{r}\varphi \text{ iff } \mathcal{M}[w], w \models \varphi & \quad \mathcal{M}, w \models \textcircled{f}\varphi \text{ iff } \mathcal{M}[-w], w \models \varphi \\ \mathcal{M}, w \models \textcircled{e}\varphi \text{ iff } \mathcal{M}[*], w \models \varphi & \quad \mathcal{M}, w \models \textcircled{k} \text{ iff } w \in S \end{array}$$

The 'remember' operator $\textcircled{r}$ (a unary modality) marks the current state as being 'already visited', by storing it in S. In contrast, the 'forget' operator $\textcircled{f}$ removes the current state from the memory, while the 'erase' operator $\textcircled{e}$ wipes out the memory. These are the operators we use to update the memory. On the other hand, the zero-ary operator $\textcircled{k}$ (for 'known') queries S to check if the current state is in the memory[5].

Besides these basic operators, we can also impose constraints on the interplay between memory storage and the standard modalities. There are some contexts when we do not need $\textcircled{r}$ and $\langle r\rangle$ as two separate operators: we are only interested in the trail of memorized points we used to evaluate a formula (for details on possible applications see [7]). In these cases the $\langle\!\langle r\rangle\!\rangle$ operator will be handy:

$$\mathcal{M}, w \models \langle\!\langle r\rangle\!\rangle\varphi \text{ iff } \exists w' \in W, wR_r w' \text{ and } \mathcal{M}[w], w' \models \varphi$$

We will denote the dual of this operator as $[\![r]\!]$, with the usual interpretation. As it was showed in [5], this operator is very useful to regain decidability for some fragments of memory logic.

A particularly interesting class of models to investigate is the class $\mathcal{C}_\emptyset$ where the memory is empty, i.e., $\mathcal{C}_\emptyset = \{\mathcal{M} \mid \mathcal{M} = \langle F, V, \emptyset\rangle\}$. It is natural to consider starting to evaluate a formula in a model of $\mathcal{C}_\emptyset$, as it is over $\mathcal{C}_\emptyset$ that the operators $\textcircled{k}$ and $\textcircled{r}$ have the most natural interpretation. As it is shown in [5], the restriction to this class has important effects on expressivity and decidability. It is worth noting that a formula is *initially* evaluated in a model of $\mathcal{C}_\emptyset$, but during the evaluation the model can change to one with nonempty memory. This dynamic behavior is a distinctive feature of memory logics over the classical modal logic: the value of S changes as the evaluation of the formula proceeds. This is not different to what happens with an assignment during the evaluation of a first order formula.

[5] Notice that all these operators are self dual.

It is well known that a classical modal model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ can be seen as first order model over an appropriate signature, and that there is a standard translation ST_x transforming every modal formula φ into a first order formula $\mathsf{ST}_x(\varphi)$ with x as its only free variable such that $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, g^x_w \models \mathsf{ST}_x(\varphi)$, where g^x_w is an arbitrary assignment that maps x to w (on the left, $\mathcal{M}$ should be considered as a first order model, and $\models$ as the standard first order satisfiability relation). Similarly, any memory model can be seen as a first order model, and we can define a translation which transforms memory formulas into equivalent first order formulas (for more details see [5]). We will use this result for some results in this article.

In [2,5] some computational aspects of memory logics were studied, together with results for separating different memory logics in terms of expressive power. In [4,5] the focus was put in proof theoretical results. In this article we analyze some important theorems of the basic model theory for memory logics. The main tool for all our results on characterization, definability and interpolation is the notion of *bisimulation*. In Section 2 we present suitable notions of bisimulation for different memory logics. In Section 3 we state a van Benthem like characterization theorem for memory logics and we study when a class of memory models is definable by a set of memory formulas, or by a single formula. In Section 4, we analyze the validity of the Craig interpolation theorem for many members of the family of memory logics. Finally, in Section 5 we discuss further work and draw some conclusions.

Notation. As we will be discussing many different logics, we introduce here some notational conventions. We call $\mathcal{ML}$ the basic modal logic, and add a superscript m to indicate the addition of a memory-set and the basic memory operators ⓡ and ⓚ. Additional operators included in the language are listed explicitly. Since we can choose to use $\langle r \rangle$ or $\langle\!\langle r \rangle\!\rangle$, we will also include the diamond explicitly in this list. For example, $\mathcal{ML}^m(\langle r \rangle, ⓔ)$ is the modal logic with the standard diamond operator extended with ⓡ, ⓚ and ⓔ. When we restrict initial evaluation of a formula to models in $\mathcal{C}_\emptyset$ we add $\emptyset$ as a subscript. For example, $\mathcal{ML}^m_\emptyset(\langle\!\langle r \rangle\!\rangle)$ is the modal logic with $\langle\!\langle r \rangle\!\rangle$ instead of $\langle r \rangle$, the operators ⓡ and ⓚ, and whose models have an initially empty memory. For the rest of the article, $\mathfrak{L}$ will stand for any memory logic $\mathcal{ML}^m(\ldots)$. For the sake of simplicity we restrict ourselves to the unimodal case. The generalization to the multimodal scenario is straightforward.

2 Bisimulations and saturated models

The concept of bisimulation has been extensively studied for many modal logics [1,6]. In the context of memory logics, bisimulations link pairs (A, w) between models, as we need to keep track not only of the current state but also of the current memory. Let $\mathcal{M}$ and $\mathcal{N}$ be two memory models. Then a bisimulation between $\mathcal{M}$ and $\mathcal{N}$ is a binary relation such that $(A, w) \sim (B, v)$ implies $A \cup \{w\} \subseteq |\mathcal{M}|$ and $B \cup \{v\} \subseteq |\mathcal{N}|$.

Bisimulations for the different memory logics can be defined modularly. Given a memory logic $\mathfrak{L}$, its bisimulation notion will be defined imposing restrictions to $\sim$ depending on the operators present in $\mathfrak{L}$. In Figure 1 we summarize the restrictions associated with each operator for models $\mathcal{M}$ and $\mathcal{N}$ with accessibility relations R and R' respectively.

always	(nontriv)	$\sim$ is not empty.
always	(agree)	If $(A, w) \sim (B, v)$, then w and v make the same propositional variables true.
ⓚ	(kagree)	If $(A, w) \sim (B, v)$, then $w \in A$ if and only if $v \in B$.
ⓡ	(remember)	If $(A, w) \sim (B, v)$, then $(A \cup \{w\}, w) \sim (B \cup \{v\}, v)$.
ⓕ	(forget)	If $(A, w) \sim (B, v)$, then $(A \setminus \{w\}, w) \sim (B \setminus \{v\}, v)$.
ⓔ	(erase)	If $(A, w) \sim (B, v)$, then $(\emptyset, w) \sim (\emptyset, v)$.
$\langle r \rangle$	(forth)	If $(A, w) \sim (B, v)$ and wRw', then there exists $n' \in \lvert\mathcal{N}\rvert$ such that $vR'v'$ and $(A, w') \sim (B, v')$.
	(back)	If $(A, w) \sim (B, v)$ and $vR'v'$, then there exists $w' \in \lvert\mathcal{M}\rvert$ such that wRw' and $(A, w') \sim (B, v')$.
$\langle\!\langle r \rangle\!\rangle$	(mforth)	If $(A, w) \sim (B, v)$ and wRw', then there exists $v' \in \lvert\mathcal{N}\rvert$ such that $vR'v'$ and $(A \cup \{w\}, w') \sim (B \cup \{v\}, v')$.
	(mback)	If $(A, w) \sim (B, v)$ and $vR'v'$, then there exists $w' \in \lvert\mathcal{M}\rvert$ such that wRw' and $(A \cup \{w\}, w') \sim (B \cup \{v\}, v')$.

Fig. 1. Operator restrictions for a modular memory bisimulation definition.

With these definitions, we have presented bisimulation notions for all memory logics introduced in Section 1.

If $\mathcal{M}$ is a model and $w \in |\mathcal{M}|$, we call the pair $\langle \mathcal{M}, w \rangle$ a *pointed model.* Given two pointed models $\langle \mathcal{M}, w \rangle$ and $\langle \mathcal{N}, v \rangle$, where $\mathcal{M} = \langle W, R, V, S \rangle$ and $\mathcal{N} = \langle W', R', V', S' \rangle$, we write $\mathcal{M}, w \,\underline{\leftrightarrow}\, \mathcal{N}, v$ if there is a bisimulation linking (S, w) and (S', v). The exact type of bisimulation involved will usually be clear from context; we will write $\underline{\leftrightarrow}_{\mathfrak{L}}$ when we need to specify that the bisimulation corresponds to the logic $\mathfrak{L}$. We write $\mathcal{M}, w \equiv_{\mathfrak{L}} \mathcal{N}, v$ when both models satisfy the same $\mathfrak{L}$-formulas, i.e., for all $\varphi \in \mathfrak{L}$, $\mathcal{M}, w \models \varphi$ iff $\mathcal{N}, v \models \varphi$. We will again drop the $_{\mathfrak{L}}$ subindex when no confusion arises.

The basic property expected from bisimulation is that they should preserve the satisfiability of formulas. The following theorem states that this is the case for the bisimulations we introduced (see [7] for details).

Theorem 1. *If* $\mathcal{M}, w \,\underline{\leftrightarrow}_{\mathfrak{L}}\, \mathcal{N}, v$ *then* $\mathcal{M}, w \equiv_{\mathfrak{L}} \mathcal{N}, v$.

With all preliminaries concerning bisimulation already introduced, we now proceed to the notion of ω-saturated models and Hennessy-Milner classes, which will lead to our first result: the class of ω-saturated models is a Hennessy-Milner class for all memory logics we introduced, with respect to the appropriate notion of bisimulation. This property will be fundamental for the results concerning characterization and definability established in the next section.

The notion of ω-saturation [8,9] is defined for first order models, but it also applies to memory models using the correspondence between memory and first order models discussed in Section 1. These models will prove to be a very useful tool. We have already seen that if two states are bisimilar, then they are modally equivalent. The converse, in general, does not hold. We say that a class $\mathcal{C}$ of models *has the Hennessy-Milner* property with respect to $\mathfrak{L}$-bisimulations (or, simply, that the class is Hennessy-Milner for $\mathfrak{L}$) if any two $\mathfrak{L}$-equivalent models in $\mathcal{C}$ are $\mathfrak{L}$-bisimilar. As we will prove in Theorem 4, ω-saturated models are Hennessy-Milner for all memory logics $\mathfrak{L}$.

But ω-saturated models have other important properties, like the '*intra-model* compactness property' enunciated below (the proof is a straightforward modification of the result in [6] for the basic modal logic).

Proposition 2. *Let $\mathcal{M} = \langle W, R, V, S\rangle$ be ω-saturated, Σ be a set of $\mathfrak{L}$-formulas and $w \in W$. If every finite subset $\Delta \subseteq \Sigma$ satisfies $\mathcal{M}, v_\Delta \models \Delta$ for some R-successor v_Δ of w then there exists v, an R-successor of w, such that $\mathcal{M}, v \models \Sigma$.*

It is also the case that ω-saturation is preserved under the operation of memorizing a finite set of elements. The proof can be found in [7].

Proposition 3. *Let $\mathcal{M}$ be ω-saturated. For any finite $A \subseteq |\mathcal{M}|$, $\mathcal{M}[A]$ is ω-saturated.*

Not all models are ω-saturated but a classic theorem of first order logic [8,9] states that every model $\mathcal{M}$ has an ω-saturated extension $\mathcal{M}^+$ with the same first order theory and, a fortiori, the same $\mathfrak{L}$ theory for any memory logic $\mathfrak{L}$. This extension is created by taking an ultrapower of the model with a special kind of ultrafilter[6].

We now prove that, for every memory logic $\mathfrak{L}$, the class of ω-saturated models has the Hennessy-Milner property with respect to $\mathfrak{L}$-bisimulations.

Theorem 4. *Let $\mathfrak{L}$ be a memory logic, the class of ω-saturated models has the Hennessy-Milner property with respect to $\mathfrak{L}$-bisimulations.*

Proof (Sketch). As we want to consider all the possible logics from the family of memory logics, we prove that, for any two ω-saturated models $\langle \mathcal{M}, w\rangle$ and $\langle \mathcal{N}, v\rangle$ such that $\mathcal{M}, w \equiv_{\mathfrak{L}} \mathcal{N}, v$ there is an $\mathfrak{L}$-bisimulation between them. We do this by considering every possible operator and show that we can construct a bisimulation that satisfies the constraints associated for that operator.

See the Appendix for full details. □

The proof of the theorem above is fairly straightforward, but the result itself is surprising in its generality and can be taken as evidence of a harmonious match between the notion of bisimulation we introduced and the general model theory of memory logics.

[6] In what follows we will assume that the reader is familiar with the definition of ultraproducts, ultrapowers and ultrafilters (consult [10] if necessary).

3 Characterization and definability

While investigating the properties of a new modal logic, a fairly standard approach is to try to characterize it as a fragment of a better known logic. A classical example of this kind of results is van Benthem's characterization of the basic modal logic as the bisimulation invariant fragment of first order logic. These type of characterizations allows for the transfer of results and for a better understanding of the logic. In the following theorem we state an analogue result for memory logics. Due to space limitations we only give a sketch of the proof along with citations that should suffice to complete it.

We say that a first order formula $\alpha(x)$ is *invariant for* $\mathfrak{L}$*-bimulations* if for all models $\mathcal{M}, \mathcal{N}$ and $w \in |\mathcal{M}|, v \in |\mathcal{N}|$ such that $\mathcal{M}, w \leftrightarrows_{\mathfrak{L}} \mathcal{N}, v$ we have $\mathcal{M}, g^x_w \models \alpha(x)$ iff $\mathcal{N}, g^x_v \models \alpha(x)$.

Theorem 5 (Characterization). *A first order formula $\alpha(x)$ (with free variable x, and in the proper signature) is equivalent to the translation of an $\mathfrak{L}$-formula iff $\alpha(x)$ is invariant for $\mathfrak{L}$-bisimulations.*

Proof (Sketch). The left to right direction is a consequence of Theorem 1. As observed in [11] the main ingredient for the right to left direction is that the class of ω-saturated models have the Hennessy-Milner property. This fact was proved true for the family of memory logics in Theorem 4. The rest of the proof is a routine rephrase of the one found in [6] for the basic modal logic. □

Notice that the result above holds for all the memory logics we introduced.

We now proceed to investigate definability. The study of definability of classes of models – i.e., given an arbitrary logic L which are the classes of models that can be captured as those satisfying a formula (or a set of formulas) of L – is well developed. Results of this kind are well known, for example, in first order logics. Traditionally, a class of models that is definable by means of a set of first order formulas is called *elementary* and those that can be defined by means of a single formula are called *basic elementary* classes.

Definability results for different modal logics have also been established [1,6]. Once more, the results for basic modal logic lifts to memory logics if we consider the appropriate notion of bisimulation.

Theorem 6 (Definability by a set). *A class of pointed models $\mathcal{C}$ is definable by a set of $\mathfrak{L}$-formulas iff $\mathcal{C}$ is closed under $\mathfrak{L}$-bisimulations and under ultraproducts; and the complement of $\mathcal{C}$ is closed under ultrapowers.*

Proof. From left to right. Suppose that $\mathcal{C}$ is defined by the set Γ of $\mathfrak{L}$-formulas and there is a model $\langle \mathcal{M}, w \rangle \in \mathcal{C}$ such that $\mathcal{M}, w \leftrightarrows \mathcal{N}, v$ for some model $\mathcal{N}, v$. As $\langle \mathcal{M}, w \rangle \in \mathcal{C}$ it must occur that $\mathcal{M}, w \models \Gamma$. By bisimulation preservation we have $\mathcal{N}, v \models \Gamma$ therefore $\langle \mathcal{N}, v \rangle \in \mathcal{C}$. Hence, $\mathcal{C}$ is closed under $\mathfrak{L}$-bisimulations.

If $\mathcal{C}$ is definable by a set Γ of $\mathfrak{L}$-formulas it is also defined by the first order translation of Γ. Therefore $\mathcal{C}$ is elementary which implies that it is closed under ultraproducts and its complement is closed under ultrapowers [8,9].

From right to left. Suppose $\mathcal{C}$ is closed under $\mathfrak{L}$-bisimulations and ultraproducts, while its complement is closed under ultrapowers. Let Γ be the set of $\mathfrak{L}$-formulas true in every model of $\mathcal{C}$. Trivially $\mathcal{C} \models \Gamma$. We still have to show that if $\mathcal{M}, w \models \Gamma$ then $\langle \mathcal{M}, w \rangle \in \mathcal{C}$. Define the following set

$$\mathsf{Th}^w(x) = \{\mathsf{ST}_x(\varphi) : \varphi \text{ is an } \mathfrak{L}\text{-formula and } \mathcal{M}, w \models \varphi\}.$$

We state that $\mathsf{Th}^w(x)$ is satisfiable in $\mathcal{C}$. For suppose not. By compactness, there is a finite subset $\Sigma_0 \subseteq \mathsf{Th}^w(x)$ such that $\Sigma_0 = \{\sigma_1, \ldots, \sigma_n\}$ is not satisfiable in $\mathcal{C}$.[7] This means that the formula $\psi = \neg \bigwedge_i \sigma_i$ is valid in $\mathcal{C}$ and therefore $\psi \in \Gamma$. This is absurd because it is obvious that $\mathcal{M}, w \not\models \psi$ and by hypothesis $\mathcal{M}, w \models \Gamma$. Hence, there is a model $\langle \mathcal{N}, v \rangle \in \mathsf{K}$ such that $\mathcal{N}, v \models \mathsf{Th}^w(x)$. It is easy to see that these models satisfy $\mathcal{N}, v \equiv_{\mathfrak{L}} \mathcal{M}, w$.

To finish, suppose that $\langle \mathcal{M}, w \rangle \notin \mathcal{C}$, we take ω-saturated extensions $\langle \mathcal{N}^*, v^* \rangle \in \mathcal{C}$ and $\langle \mathcal{M}^*, w^* \rangle \notin \mathcal{C}$. As ω-saturated models have the Hennessy-Milner property (by Theorem 4) this implies that $\mathcal{N}^*, v^* \leftrightarrows_{\mathfrak{L}} \mathcal{M}^*, w^*$. As $\mathcal{C}$ is closed under bisimulations then $\langle \mathcal{M}, w \rangle \in \mathcal{C}$. Absurd, therefore $\langle \mathcal{M}, w \rangle$ must be in $\mathcal{C}$. □

Theorem 7 (Definability by a single formula). *A class of pointed models $\mathcal{C}$ is definable by a single $\mathfrak{L}$-formula iff $\mathcal{C}$ is closed under $\mathfrak{L}$-bisimulations and both $\mathcal{C}$ and its complement are closed under ultraproducts.*

Proof. From left to right. Suppose $\mathcal{C}$ is definable by a single $\mathfrak{L}$-formula φ. Observe that the complement of $\mathcal{C}$ is defined by $\neg\varphi$. Using Theorem 6 on $\mathcal{C}$ with $\Gamma = \{\varphi\}$ and on its complement with $\Gamma = \{\neg\varphi\}$ we conclude what we wanted to prove.

From right to left. Suppose that $\mathcal{C}$ is closed under $\mathfrak{L}$-bisimulations and both $\mathcal{C}$ and its complement are closed under ultraproducts. As the bisimulation relation is symmetric it is easy to see that $\mathcal{C}$ is closed under bisimulations iff its complement is. Using this fact and Theorem 6 twice we have sets of formulas Γ_1 defining $\mathcal{C}$ and Γ_2 defining its complement.

It is obvious that the union of these sets cannot be consistent. Therefore, by compactness, there exist $\{\alpha_1, \ldots, \alpha_n\} \subseteq \Gamma_1$ and $\{\beta_1, \ldots, \beta_m\} \subseteq \Gamma_2$ such that $\bigwedge_i \alpha_i \to \neg \bigwedge_j \beta_j$ is valid. We claim that it is exactly $\varphi = \bigwedge_i \alpha_i$ that defines $\mathcal{C}$.

If $\langle \mathcal{M}, w \rangle \in \mathcal{C}$, it satisfies Γ_1 and in particular φ. Suppose that $\mathcal{M}, w \models \varphi$. Hence $\mathcal{M}, w \models \neg \bigwedge_j \beta_j$ and therefore $\mathcal{M}, w \not\models \Gamma_2$; i.e., $\langle \mathcal{M}, w \rangle \in \mathcal{C}$. □

4 Interpolation

The notion of bisimulation also plays a crucial role for proving and disproving interpolation properties. Given a formula φ, let $\mathsf{props}(\varphi)$ be the set of propositional symbols occurring in φ. A modal logic has *interpolation over propositional symbols* on a class $\mathcal{C}$, if for all formulas φ, ψ such that $\mathcal{C} \models \varphi \to \psi$, there is a modal formula δ (usually called the *interpolant*) such that $\mathcal{C} \models \varphi \to \delta$, $\mathcal{C} \models \delta \to \psi$, and $\mathsf{props}(\delta) \subseteq \mathsf{props}(\varphi) \cap \mathsf{props}(\psi)$. Note that there is no restriction on the modalities occurring in δ.

[7] The compactness theorem preserves ultraproducts-closed classes (see [11]).

We will show that most of the memory logics we are studying lack interpolation (with the exception of $\mathcal{ML}^m(\langle\!\langle r \rangle\!\rangle)$ and $\mathcal{ML}^m(\langle\!\langle r \rangle\!\rangle, \textcircled{f})$). We will use a classic technique to prove this, whose general schema is the following. First, we define φ and ψ such that $\varphi \rightarrow \psi$ is a valid formula. Then, we find two models $\langle \mathcal{M}, w \rangle$ and $\langle \mathcal{M}', w' \rangle$, such that w and w' are bisimilar in the common language of φ and ψ, but $\mathcal{M}, w \models \varphi$ while $\mathcal{M}', w' \models \neg\psi$. This is enough to claim that interpolation fails. For suppose that interpolation holds. Then there is an interpolant δ in the common language of φ and ψ such that $\varphi \rightarrow \delta$ and $\delta \rightarrow \psi$ are valid. Therefore δ holds at $\langle \mathcal{M}, w \rangle$. Because w and w' are bisimilar in the common language, δ also holds at $\langle \mathcal{M}', w' \rangle$. This implies that ψ holds at $\langle \mathcal{M}', w' \rangle$ too, but this is a contradiction, since we assumed that $\neg\psi$ holds there.

In the context of memory logics, there is a choice to make concerning the inclusion of $\textcircled{k}$ in the common language. We will use the term *interpolation over propositional symbols and known* when we decide to include it in the common language. We will just say "the common language" when no confusion between the two notions can arise. Observe that $\top$ can always occur in the interpolant, since otherwise the definition of interpolation can be easily trivialized. Finally, unless we explicitly say otherwise, we prove interpolation (or the lack thereof) for the class of all models.

We start by showing that interpolation over propositional symbols fails for $\mathcal{ML}^m(\langle r \rangle)$ and its extension with the $\textcircled{e}$ operator. This is also true for some fragments that use $\langle\!\langle r \rangle\!\rangle$ instead of $\langle r \rangle$, both over the class of all models and over $\mathcal{C}_\emptyset$.

Theorem 8. *The logics $\mathcal{ML}^m(\langle r \rangle)$, $\mathcal{ML}^m(\langle r \rangle, \textcircled{e})$, $\mathcal{ML}^m_\emptyset(\langle r \rangle)$, $\mathcal{ML}^m_\emptyset(\langle\!\langle r \rangle\!\rangle)$ and $\mathcal{ML}^m_\emptyset(\langle r \rangle, \textcircled{e})$ lack interpolation over propositional symbols.*

Proof (Sketch). Full details are given in the Appendix. The key ingredient of the proofs for each logic is the ability to find two models which are bisimilar in the common language. These results are strongly based on bisimilar models used in [5] to investigate relative expressive power of different memory logics. □

We leave the analysis for $\textcircled{f}$ open, since we could not find an equivalent pair of models for this case. See [7] for more details.

Now we show that $\mathcal{ML}^m(\langle\!\langle r \rangle\!\rangle)$ has interpolation over propositional symbols and known with respect to a quite general class of models. The technique we use here is similar to the one presented in [12]. To develop the proof we will need some tools from model theory. We introduce some definitions and preliminary results and refer the reader to [8,9,12,13] for details.

Throughout the rest of this section, $\leftrightarroweq$ refers to $\mathcal{ML}^m(\langle\!\langle r \rangle\!\rangle)$-bisimulation. We will use $\leftrightarroweq_{\mathcal{ML}}$ when we want to refer to $\mathcal{ML}$-bisimulations.

Definition 9. *1. A* total $\mathcal{ML}$ frame bisimulation *between frames $\langle W, R \rangle$ and $\langle W', R' \rangle$ is a total binary relation on $W \times W'$ satisfying conditions (nontriv), (forth) and (back) from Figure 1.*

2. An $\mathcal{ML}$-bisimulation product *of a set of frames $\{\mathcal{F}_i \mid i \in I\}$ is a subframe $\mathcal{B}$ of the cartesian product $\Pi_i \mathcal{F}_i$ such that for each $i \in I$, the natural projection function $f_i : \mathcal{B} \rightarrow \mathcal{F}_i$ is a surjective bounded morphism.*

Bisimulation producs, together with the following theorem (see [13] for the proof), allow us to construct a new frame using a total $\mathcal{ML}$ frame bisimulation between two given frames. This will be helpful later to construct a model that will act as a witness for the interpolant.

Theorem 10. *Let $\mathcal{H}$ be a subframe of the product $\mathcal{F} \times \mathcal{G}$. Then $\mathcal{H}$ is an $\mathcal{ML}$-bisimulation product of $\mathcal{F}$ and $\mathcal{G}$ iff the domain of $\mathcal{H}$ is a total $\mathcal{ML}$ frame bisimulation between $\mathcal{F}$ and $\mathcal{G}$.*

The last ingredient we need is to define total bisimulation in the context of memory logics. Intuitively, it is a bisimulation in which every possible relevant pairs are related.

Definition 11 (Total $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-bisimulation). *Let $\mathcal{M} = \langle W, R, V, S\rangle$ and $\mathcal{N} = \langle W', R', V', S'\rangle$ be two models of $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$. We say that $\mathcal{M}, w$ and $\mathcal{N}, v$ are* totally bisimilar *($\mathcal{M}, w \leftrightarrows^T \mathcal{N}, v$) when there is bisimulation $\sim$ between $\mathcal{M}, w$ and $\mathcal{N}, v$ and*

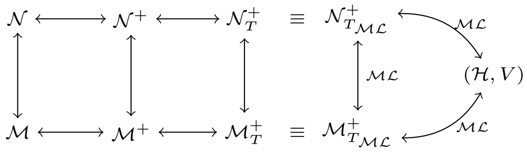
1. *for every $A = \{a_1, \ldots, a_k\} \subseteq W$ with $a_i R a_{i+1}$, and every $a \in W$ there is a $B = \{b_1, \ldots, b_k\} \subseteq W'$ with $b_i R' b_{i+1}$ and $b \in W'$ such that $(A, a) \sim (B, b)$*
2. *for every $B = \{b_1, \ldots, b_k\} \subseteq W'$, and every $b \in W'$ there is $A = \{a_1, \ldots, a_k\} \subseteq W$ with $a_i R a_{i+1}$ and $a \in M$ such that $(A, a) \sim (B, b)$.*

Theorem 12. *Let $\mathcal{C}$ be any elementary frame class closed under generated subframes and bisimulation products. Then $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$ and $\mathcal{ML}^m(\langle\langle r \rangle\rangle, ⓕ)$ have interpolation over propositions and known relative to the class of all models with frame in $\mathcal{C}$.*

Proof (Sketch). Full details are provided in the Appendix. Suppose there are two $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-formulas φ and ψ such that $\varphi \to \psi$ is valid, but it does not have an interpolant in the common language. In general, the bisimulations we discuss here between a pair of models are always established with respect to the common language of φ and ψ. We first show that there are two models $\mathcal{M}$ and $\mathcal{N}$ such that $\mathcal{M}, w \models \varphi$ and $\mathcal{N}, v \models \neg\psi$. We next take ω-saturated models $\mathcal{M}^+$ and $\mathcal{N}^+$ of $\mathcal{M}$ and $\mathcal{N}$ respectively and show $\mathcal{M}^+, w \leftrightarrows^T \mathcal{N}^+, v$. According to the tree model property for $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$ (see [5]), we take equivalent tree $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-models $\mathcal{M}^+_T$ and $\mathcal{N}^+_T$ such that $\mathcal{M}^+, w \leftrightarrows^T \mathcal{M}^+_T, w$ and $\mathcal{N}^+, v \leftrightarrows^T \mathcal{N}^+_T, v$. We conclude $\mathcal{M}^+_T, w \leftrightarrows^T \mathcal{N}^+_T, v$.

Then we switch to the basic modal logic $\mathcal{ML}$. Let $\mathcal{M}^+_{T_{\mathcal{ML}}}$ and $\mathcal{N}^+_{T_{\mathcal{ML}}}$ be the corresponding $\mathcal{ML}$-models of $\mathcal{M}^+_T$ and $\mathcal{N}^+_T$ respectively (shifting the signature to $\langle \text{PROP} \cup \{known\}, \text{REL}\rangle$). Being $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$ an extension of $\mathcal{ML}$, we have $\mathcal{M}^+_{T_{\mathcal{ML}}} \leftrightarrows^T_{\mathcal{ML}} \mathcal{N}^+_{T_{\mathcal{ML}}}$. Using Theorem 10, one can show that there is a bisimulation product $\mathcal{H} \in \mathcal{C}$ of the frames of $\mathcal{M}'$ and $\mathcal{N}'$, and a valuation V such that $(\mathcal{H}, V), \langle w, v\rangle \models (\varphi \wedge \neg\psi)[ⓚ/known]$.

Since by its definition, $\mathcal{H}$ is a tree, we can return to $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$ and conclude that $\varphi \wedge \neg\psi$ is satisfiable in some $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-model based on a frame in $\mathcal{C}$, contradicting our hypothesis. Graphically, the general schema is the following (the double headed arrows represent total bisimulations):



Following this schema the result can be proved for $\mathcal{ML}^m(\langle\langle r\rangle\rangle)$. Then, interpolation for $\mathcal{ML}^m(\langle\langle r\rangle\rangle, \textcircled{f})$ is straightforward using the equivalence preserving translations defined in [5] between $\mathcal{ML}^m(\langle\langle r\rangle\rangle, \textcircled{f})$ and $\mathcal{ML}^m(\langle\langle r\rangle\rangle)$. □

5 Conclusions and further work

In this article we investigated some model theoretical properties of several memory logics. Fist we analyzed memory logics in terms of first-order characterization and definability. These properties hold for all the logics we introduced, thanks to a general Hennessy-Milner property for ω-saturated models. Then we studied interpolation and showed that the property fails for many memory logics both over the class of all models and over $\mathcal{C}_\emptyset$. On the other hand, we stablished interpolation over propositional symbols and known for $\mathcal{ML}^m(\langle\langle r\rangle\rangle)$ and $\mathcal{ML}^m(\langle\langle r\rangle\rangle, \textcircled{f})$ over many different classes of models. Bisimulations were a key tool to tackle these problems. The results presented here help complete a picture of the properties of memory logics and contributes to understanding what they are, how they behave, and which is their relation with other well-known logics.

There are some pending problems that are worth investigating. The expressive power of some memory logics is still not well understood. This is the case of the $\textcircled{f}$ operator and its combination with other memory and modal operators. The open questions about this operator mentioned in [5] are still unsolved, and they result in related open issues concerning interpolation.

Finally, the Beth definability property is usually studied together with interpolation, and for many logics, a proof of the former can be obtained once a proof of the later is at hand. Both properties are closely connected, and the logics having one but not the other are relatively few (see [14] for examples). Alas! the case for memory logics is not that simple. Even though some weak results concerning Beth definability for memory logics have been established (see [7]), a general conclusion is still missing.

References

1. Blackburn, P., Wolter, F., van Benthem, J., eds.: Handbook of Modal Logics. Elsevier (2006)
2. Areces, C., Figueira, D., Figueira, S., Mera, S.: Expressive power and decidability for memory logics. In: Logic, Language, Information and Computation. Volume 5110 of Lecture Notes in Computer Science., Springer Berlin / Heidelberg (2008) 56–68 Proceedings of WoLLIC 2008.

3. Areces, C., Figueira, S., Mera, S.: Completeness results for memory logics. In: Proceedings of the Symposium on Logical Foundations of Computer Science. Volume 5407 of LNCS., Springer-Verlag (2009) 16–30
4. Areces, C., Figueira, D., Gorín, D., Mera, S.: Tableaux and model checking for memory logics. In: Proceedings of Tableaux09. (2009) 47–61
5. Areces, C., Figueira, D., Figueira, S., Mera, S.: The expressive power of memory logics. Review of Symbolic Logic (2010) In press (pre-print version in `http://www.glyc.dc.uba.ar/sergio/papers/rsl.pdf`).
6. Blackburn, P., de Rijke, M., Venema, Y.: Modal Logic. Cambridge University Press (2001)
7. Mera, S.: Modal Memory Logics. PhD thesis, Universidad de Buenos Aires & Université Henri Poincare, Buenos Aires, Argentina (2009)
8. Chang, C., Keisler, H.: Model Theory. Third edn. Volume 73 of Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co. (1990)
9. Doets, K.: Basic model theory. University of Chicago Press (1996)
10. Keisler, H.J.: The ultraproduct construction. In: Proceedings of the Ultramath Conference, Pisa, Italy (2008)
11. Carreiro, F.: Characterization and definability in modal first-order fragments. Master's thesis, Universidad de Buenos Aires (2010) `arXiv:1011.4718`.
12. ten Cate, B.: Model theory for extended modal languages. PhD thesis, University of Amsterdam (2005) ILLC Publications, Ph. D. Dissertation series, Amsterdam.
13. Marx, M., Venema, Y.: Multi-dimensional modal logic. Kluwer (1997)
14. Hoogland, E.: Definability and interpolation: Model-theoretic investigations. PhD thesis, ILLC. Universiteit van Amsterdam (2001)

Appendix

Theorem 4. *Let $\mathfrak{L}$ be a memory logic, the class of ω-saturated models has the Hennessy-Milner property with respect to $\mathfrak{L}$-bisimulations.*

Proof. Given two ω-saturated models $\mathcal{M} = \langle W, R, V, S\rangle$ and $\mathcal{N} = \langle W', R', V', S'\rangle$ we propose the binary relation $\sim$ defined as

$$(A, w) \sim (B, v) \text{ iff } \mathcal{M}', w \equiv_{\mathfrak{L}} \mathcal{N}', v$$

as a candidate for a bisimulation where $\mathcal{M}' = \langle W, R, V, A\rangle$, $\mathcal{N}' = \langle W', R', V', B\rangle$ and $A \cup \{w\} \subseteq W$, $B \cup \{v\} \subseteq W'$. Suppose that $(A, w) \sim (B, v)$. $\sim$ satisfies *(nontriv)* and *(agree)* by definition.

(kagree): If ⓚ is an operator of $\mathfrak{L}$, then $w \in A$ iff $\mathcal{M}', w \models$ ⓚ iff $\mathcal{N}', v \models$ ⓚ iff $v \in B$. This proves that *(kagree)* is satisfied.

(remember): Suppose that ⓡ is an operator of $\mathfrak{L}$. Then $(A, w) \sim (B, v)$ implies that for every φ, $\mathcal{M}', w \models \varphi$ iff $\mathcal{N}', v \models \varphi$. In particular, $\mathcal{M}', w \models$ ⓡψ iff $\mathcal{N}', v \models$ ⓡψ which by satisfaction definition holds precisely when $\mathcal{M}'[w], w \models \psi$ iff $\mathcal{N}'[v], v \models \psi$ and hence $(A \cup \{w\}, w) \sim (B \cup \{v\}, v)$. This proves that *(remember)* is satisfied. The conditions *(forget)* and *(erase)* are stablished similarly in logics with the ⓕ and ⓔ operators.

(forth) and *(back)*: These properties are proved as for basic modal logic (see [6]).

(mforth) and *(mback)*: Since $(A, w) \sim (B, v)$, we have already seen in the $\textcircled{r}$ case that $\mathcal{M}'[w], w \models \psi$ iff $\mathcal{N}'[v], v \models \psi$.[8] This implies that $\mathcal{M}'[w], w \equiv_{\mathfrak{L}} \mathcal{N}'[v], v$. Using Lemma 3 we also know that $\langle \mathcal{M}'[w], w\rangle$ and $\langle \mathcal{N}'[v], v\rangle$ are both ω-saturated.

Suppose that w' is a successor of w. Let Σ be the set of all the formulas true at $\mathcal{M}'[w], w'$. For every finite subset $\Delta \subseteq \Sigma$ we have $\mathcal{M}'[w], w' \models \bigwedge \Delta$ and therefore $\mathcal{M}'[w], w \models \langle\!\langle r \rangle\!\rangle \bigwedge \Delta$. By $\mathfrak{L}$-equivalence we have $\mathcal{N}'[v], v \models \langle\!\langle r \rangle\!\rangle \bigwedge \Delta$ which means that for every Δ we have a v-successor which satisfies it. By Lemma 2 we can conclude that there exists v' a v-successor so that $\mathcal{N}'[v], v' \models \Sigma$.

As $\mathcal{M}'[w], w'$ and $\mathcal{N}'[v], v'$ make the same formulas true, then they are $\mathfrak{L}$-equivalent and by definition they will be related by the bisimulation. This proves that *(mforth)* is satisfied because $(A \cup \{w\}, w') \sim (B \cup \{v\}, v')$. The proof for *(mback)* is similar but switching the models. □

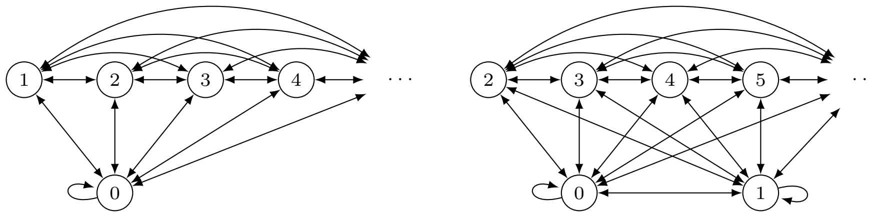
Theorem 8. *The logics $\mathcal{ML}^m(\langle r\rangle)$, $\mathcal{ML}^m(\langle r\rangle, \textcircled{e})$, $\mathcal{ML}^m_\emptyset(\langle r\rangle)$, $\mathcal{ML}^m_\emptyset(\langle\!\langle r\rangle\!\rangle)$ and $\mathcal{ML}^m_\emptyset(\langle r\rangle, \textcircled{e})$ lack interpolation over propositional symbols.*

Proof. We show each case separately.

$\mathcal{ML}^m_\emptyset(\langle r\rangle)$: Let $\varphi = q \wedge \textcircled{r}[r](\neg\textcircled{k} \rightarrow \varphi')$. If $\mathcal{M}, w \models \varphi$ then q is true at w and any successor of w different from w satisfies φ'. Now, let $\varphi' = \neg q \wedge \neg\textcircled{r}\langle r\rangle(\textcircled{k} \wedge \neg q)$. With this definition of φ', if $\mathcal{M}, w \models \varphi$ then for all v such that wRv and $v \neq w$ we have $\neg vRv$.

Let $\psi = p \wedge \langle r\rangle(\neg p \wedge \textcircled{r}\langle r\rangle\textcircled{k})$. If $\mathcal{M}, w \models \psi$ then there is $v \neq w$ such that wRv and vRv. It is clear that $\varphi \wedge \psi$ is a contradiction, so $\varphi \rightarrow \neg\psi$ is valid.

Let $\mathcal{M}_1 = \langle \mathbb{N}, R_1, \emptyset, \emptyset\rangle$ and $\mathcal{M}_2 = \langle \mathbb{N}, R_2, \emptyset, \emptyset\rangle$, where $R_1 = \{(n, m) \mid n \neq m\} \cup \{(0, 0)\}$ and $R_2 = R_1 \cup \{(1, 1)\}$. Graphically,



$\mathcal{M}_1$ $\mathcal{M}_2$

where the accessibility relation is the transitive closure of the arrows shown but without reflexive loops excepts those explicitly marked. In [5] it was shown that $\langle \mathcal{M}_1, 0\rangle$ and $\langle \mathcal{M}_2, 0\rangle$ are bisimilar over $\mathcal{ML}^m_\emptyset(\langle r\rangle)$. Now, define the models $\mathcal{M}'_1$ and $\mathcal{M}'_2$ as $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively but with a nonempty valuation in the following way: $\mathcal{M}'_1 = \langle \mathbb{N}, R_1, V_1, \emptyset\rangle$ and $\mathcal{M}'_2 = \langle \mathbb{N}, R_2, V_2, \emptyset\rangle$, where $R_1 = \{(n, m) \mid n \neq m\} \cup \{(0, 0)\}$, $R_2 = R_1 \cup \{(1, 1)\}$, $V_1(q) = \{0\}$ and $V_2(p) = \{0\}$. One can verify that $\langle \mathcal{M}'_1, 0\rangle$ and $\langle \mathcal{M}'_2, 0\rangle$ are bisimilar over the common language and that $\mathcal{M}'_1, 0 \models \varphi$ and $\mathcal{M}'_2, 0 \models \psi$.

Suppose there is an interpolant χ over the common language of φ and ψ for the valid formula $\varphi \rightarrow \neg\psi$. On the one hand, since φ is true at $\langle \mathcal{M}'_1, 0\rangle$ then χ

[8] We can use $\textcircled{r}$ here because we required that every memory logic should have it.

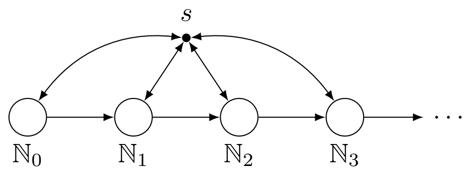
also is. On the other, since ψ is true at $\langle \mathcal{M}_2', 0\rangle$ then $\neg\chi$ also is. Then we have that $\mathcal{M}_1', 0 \models \chi$ and $\mathcal{M}_2', 0 \models \neg\chi$, which is a contradiction because $\langle \mathcal{M}_1', 0\rangle$ and $\langle \mathcal{M}_2', 0\rangle$ are bisimilar over the common language.

$\mathcal{ML}^m(\langle r\rangle)$: Let φ and ψ be as in the case for $\mathcal{ML}^m_\emptyset(\langle r\rangle)$. Let $\theta = \neg\textcircled{k} \wedge [r]\neg\textcircled{k} \wedge [r][r]\neg\textcircled{k}$. Define $\varphi' = \varphi \wedge \theta$ and $\psi' = \psi \wedge \theta$ and repeat the proof above.

$\mathcal{ML}^m_\emptyset(\langle\!\langle r\rangle\!\rangle)$: Observe that in the proof for $\mathcal{ML}^m_\emptyset(\langle r\rangle)$, instead of ψ, one could use $\psi' = p \wedge \textcircled{r}\langle r\rangle(\neg p \wedge \textcircled{r}\langle r\rangle(\textcircled{k} \wedge \neg p))$. Now, in both φ and ψ', all occurrences of $\langle r\rangle$ are of the form $\textcircled{r}\langle r\rangle$, and all occurrences of $[r]$ are of the form $\textcircled{r}[r]$. Therefore they can be translated to $\langle\!\langle r\rangle\!\rangle$ and $[\![r]\!]$ preserving equivalence. Since $\mathcal{ML}^m_\emptyset(\langle\!\langle r\rangle\!\rangle)$ is less expressive than $\mathcal{ML}^m_\emptyset(\langle r\rangle)$, both models of the proof for $\mathcal{ML}^m_\emptyset(\langle r\rangle)$ are $\mathcal{ML}^m_\emptyset(\langle\!\langle r\rangle\!\rangle)$-bisimilar and therefore the argument is valid.

$\mathcal{ML}^m_\emptyset(\langle r\rangle, \textcircled{e})$: Let $\theta(q) = \textcircled{r}\langle r\rangle(q \wedge \textcircled{k} \wedge \neg\langle r\rangle(\neg q \wedge \textcircled{k}))$. Suppose $\mathcal{M}$ is a model with $S = \{w\}$ where $\mathcal{M}, w \models q$ and $\mathcal{M}, v \models \neg q$. It is not difficult to see that $\mathcal{M}, v \models \theta(q)$ iff vRw and $\neg wRv$. Now, let $\varphi = q \wedge \textcircled{r}\langle r\rangle\langle r\rangle(\neg q \wedge \theta(q))$ and $\psi = p \wedge \textcircled{r}[r][r](\neg\textcircled{k} \to (\neg p \wedge \neg\theta(p)))$ (here $\theta(p)$ is the result of replacing all occurrences of q by p in the formula $\theta(q)$). If φ is true at a point w then there are points u and $v \neq w$ such that $wRuRv$ and vRw and $\neg wRv$. If ψ is true at a point w then for all points u and $v \neq w$ such that $wRuRv$ it is not the case that and vRw and $\neg wRv$. Hence $\models \varphi \to \neg\psi$.

Let $\mathcal{M} = \langle \{s\} \cup \mathbb{N}_0 \cup \mathbb{N}_1 \cup \ldots, R, \emptyset, \emptyset\rangle$, where each $\mathbb{N}_i$ is a different copy of $\mathbb{N}$, and $R = \{(n,m) \mid n \in \mathbb{N}_i, m \in \mathbb{N}_j, i \leq j\} \cup \{(n,s),(s,n) \mid \text{for all } n \neq s\}$. Graphically,



In [5] it was showed that $\langle \mathcal{M}, w_0\rangle$ and $\langle \mathcal{M}, w_1\rangle$ are $\mathcal{ML}^m_\emptyset(\langle r\rangle, \textcircled{e})$-bisimilar, where $w_0 \in \mathbb{N}_0$ and $w_1 \in \mathbb{N}_1$. Let $\mathcal{M}'$ be as $\mathcal{M}$ but with a nonempty valuation: $V(p) = \{w_0\}$, $V(q) = \{w_1\}$, and $V(r) = \emptyset$ for all $r \in \textsc{prop}$ different from p and q. It is straightforward to verify that $\mathcal{M}', w_0 \models \psi$ and $\mathcal{M}', w_1 \models \varphi$, but $\langle \mathcal{M}', w_0\rangle$ and $\langle \mathcal{M}', w_1\rangle$ are $\mathcal{ML}^m_\emptyset(\langle r\rangle, \textcircled{e})$-bisimilar in the common language.

$\mathcal{ML}^m(\langle r\rangle, \textcircled{e})$: Let φ and ψ be as in the proof for $\mathcal{ML}^m_\emptyset(\langle r\rangle, \textcircled{e})$. It is easy to see that $\textcircled{e}\varphi \to \neg\textcircled{e}\psi$ is a valid formula in the class of $\mathcal{ML}^m(\langle r\rangle, \textcircled{e})$-models. The rest of the argument is similar. ☐

Theorem 12. *Let $\mathcal{C}$ be any elementary frame class closed under generated subframes and bisimulation products. Then $\mathcal{ML}^m(\langle\!\langle r\rangle\!\rangle)$ and $\mathcal{ML}^m(\langle\!\langle r\rangle\!\rangle, \textcircled{f})$ have interpolation over propositions and known relative to the class of all models with frame in $\mathcal{C}$.*

Proof. We only give the proof of the main theorem. The proofs for the auxiliary lemmas can be found in [7]. Let φ and ψ such that $\mathcal{C} \models \varphi \to \psi$ and let $\mathcal{L}$ be the common language of φ and ψ. Suppose for the sake of contradiction that there is no interpolant of φ and ψ in the language $\mathcal{L}$. We first state two easy lemmas:

Lemma 13. *There is a model $\mathcal{M}$ based on a frame in $\mathcal{C}$, with a state w, such that $\mathcal{M}, w \models \{\chi \mid \mathcal{C} \models \varphi \to \chi$ and $\chi \in \mathcal{L}\} \cup \{\neg\psi\}$.*

Since $\mathcal{C}$ is closed under generated subframes we may assume that $\mathcal{M}$ is generated by w.

Lemma 14. *There is a model $\mathcal{N}$ based on a frame in $\mathcal{C}$, with a state v, such that $\mathcal{N}, v \models \{\chi \mid \mathcal{M}, w \models \chi$ and $\chi \in \mathcal{L}\} \cup \{\varphi\}$.*

Again we may assume that $\mathcal{N}$ is generated by v. Let $\mathcal{M}^+$ and $\mathcal{N}^+$ be ω-saturated elementary extensions of $\mathcal{M}$ and $\mathcal{N}$ respectively. Let us suppose that the first order models $\mathcal{M}^+$ and $\mathcal{N}^+$ have domains M and N and binary relations R_1 and R_2 for the modal operator $\langle r \rangle$, respectively.

We define the relation $\sim$ between $\wp(M) \times M$ and $\wp(N) \times N$ in the following way: for all finite $A \subseteq M$ and finite $B \subseteq N$,

$$(A, a) \sim (B, b) \text{ iff for all formulas } \chi \text{ in } \mathcal{L},\ \mathcal{M}^+[A], a \models \chi \text{ iff } \mathcal{N}^+[B], b \models \chi.$$

By construction $(\emptyset, w) \sim (\emptyset, v)$. We prove that $\sim$ is a bisimulation. Call ST_x the translation from $\mathcal{ML}^m(\langle r \rangle)$ formulas to first order logic formulas defined in [5].

Lemma 15. *$\sim$ is an $\mathcal{ML}^m(\langle\!\langle r \rangle\!\rangle)$-bisimulation between $\mathcal{M}^+$ and $\mathcal{N}^+$ with respect to $\mathcal{L}$.*

Proof. By the definition of $\sim$, it is clear that the condition (*agree*) is satisfied, restricted to $\mathcal{L}$. Let us see (*mzig*). Suppose $(A, a) \sim (B, b)$ and aR_1a'. Let

$$\Gamma = \{\mathsf{ST}_x(\chi) \mid \mathcal{M}^+[A \cup \{a\}], a' \models \chi \text{ and } \chi \in \mathcal{L}\}.$$

Let c_b be a new constant denoting the element b of $\mathcal{N}^+$. We next show that $\Gamma \cup \{R(c_b, x)\}$ is realized in $\mathcal{N}^+[B \cup \{b\}]$, where R is the first order binary relation symbol for $\langle\!\langle r \rangle\!\rangle$. Since, by Lemma 3, the expansion of $\mathcal{N}^+[B \cup \{b\}]$ with the constant c_b is 1-saturated, it suffices to show that every finite subset of Γ is realized in $\mathcal{N}^+[B \cup \{b\}]$ by an R_2-successor of b. Let $\mathsf{ST}_x(\chi_1), \ldots, \mathsf{ST}_x(\chi_n) \in \Gamma$. We have $\mathcal{M}^+[A], a \models \langle\!\langle r \rangle\!\rangle(\chi_1 \wedge \cdots \wedge \chi_n)$, and therefore $\mathcal{N}^+[B], b \models \langle\!\langle r \rangle\!\rangle(\chi_1 \wedge \cdots \wedge \chi_n)$, which implies that there is an R_2-successor of b which satisfies $\chi_1 \wedge \cdots \wedge \chi_n$. I.e., in $\mathcal{N}^+[B \cup \{b\}]$ there is an R_2-successor which realizes $\{\mathsf{ST}_x(\chi_1), \ldots, \mathsf{ST}_x(\chi_n)\}$. Hence, there is b', bR_2b' such that $\mathcal{N}^+[B \cup \{b\}], g^x_{b'} \models \Gamma$. Therefore for every χ of $\mathcal{L}$, if $\mathcal{M}^+[A \cup \{a\}], a' \models \chi$ then $\mathcal{N}^+[B \cup \{b\}], b' \models \chi$. To see the other implication, suppose by contradiction that $\mathcal{N}^+[B \cup \{b\}], b' \models \chi$ but $\mathcal{M}^+[A \cup \{a\}], a' \not\models \chi$ (the case $\mathcal{M}^+[A \cup \{a\}], a' \models \chi$ but $\mathcal{N}^+[B \cup \{b\}], b' \not\models \chi$ is similar). This would imply that $\mathcal{M}^+[A \cup \{a\}], a' \models \neg\chi$ and hence $\mathcal{N}^+[B \cup \{b\}], b' \models \neg\chi$ which leads to a contradiction. The (*mzag*) condition is similar.

In order to check (*remember*), suppose that $\mathcal{M}^+[A], a \models \chi$ iff $\mathcal{N}^+[B], b \models \chi$ for all χ of $\mathcal{L}$. Now, let χ be any formula of $\mathcal{L}$. By hypothesis, $\mathcal{M}^+[A], a \models ⓡ\chi$ iff $\mathcal{N}^+[B], b \models ⓡ\chi$. Applying the definition of ⓡ, we obtain $\mathcal{M}^+[A \cup \{a\}], a \models \chi$ iff $\mathcal{N}^+[B \cup \{b\}], b \models \chi$. □

The following lemma helps prove that $\sim$ is total.

Lemma 16. *For every $a \in M$ there is $b \in N$ such that $(\emptyset, a) \sim (\emptyset, b)$; also for every $b \in N$ there is $a \in M$ such that $(\emptyset, a) \sim (\emptyset, b)$*

Corollary 17. *The $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-bisimulation $\sim$ is total.*

Applying the tree model property for $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$ (see [5]), let $\mathcal{M}^+_T$ and $\mathcal{N}^+_T$ be tree $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-models such that $\mathcal{M}^+, w \leftrightarrows^T \mathcal{M}^+_T, w$ and $\mathcal{N}^+, v \leftrightarrows^T \mathcal{N}^+_T, v$. By Corollary 17, $\mathcal{M}^+, w \leftrightarrows^T \mathcal{N}^+, v$, and by transitivity of total bisimulations, we conclude $\mathcal{M}^+_T, w \leftrightarrows^T \mathcal{N}^+_T, v$.

Now, let $\mathcal{M}^+_{T_{\mathcal{ML}}}$ and $\mathcal{N}^+_{T_{\mathcal{ML}}}$ be the $\mathcal{ML}$ equivalent models for $\mathcal{M}^+_T$ and $\mathcal{N}^+_T$. Since $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-bisimulation implies $\mathcal{ML}$-bisimulation, $\mathcal{M}^+_{T_{\mathcal{ML}}} \leftrightarrows^T_{\mathcal{ML}} \mathcal{N}^+_{T_{\mathcal{ML}}}$. Let $\mathcal{F}$ and $\mathcal{G}$ be the underlying frames of $\mathcal{M}^+_{T_{\mathcal{ML}}}$ and $\mathcal{N}^+_{T_{\mathcal{ML}}}$ respectively. Using Theorem 10, we know there is a bisimulation product $\mathcal{H} \in \mathcal{C}$ of $\mathcal{F}$ and $\mathcal{G}$ of which the domain is $\sim$. By the definition of bisimulation products, the natural projections $f : \mathcal{H} \to \mathcal{F}$ and $g : \mathcal{H} \to \mathcal{G}$ are surjective bounded morphisms. For any proposition letter $p \in \mathsf{props}(\varphi)$, let $V(p) = \{u \mid \mathcal{M}_{T_{\mathcal{ML}}}, f(u) \models p\}$, and for any proposition letter $p \in \mathsf{props}(\psi)$, let $V(p) = \{u \mid \mathcal{N}^+_{T_{\mathcal{ML}}}, g(u) \models p\}$. The properties of $\sim$ guarantee that this V is well-defined for $p \in \mathsf{props}(\varphi) \cap \mathsf{props}(\psi)$. By a standard argument, the graph of f is a bisimulation between $(\mathcal{H}, V)$ and $\mathcal{M}^+_{T_{\mathcal{ML}}}$ with respect to $\mathsf{props}(\varphi)$, and the graph of g is a bisimulation between $(\mathcal{H}, V)$ and $\mathcal{N}^+_{T_{\mathcal{ML}}}$ with respect to $\mathsf{props}(\psi)$.

Now we have the appropriate model in which the contradiction is made explicit, but we have to be able to raise this result to $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$. Notice that the model $(\mathcal{H}, V)$ is a tree, since it is the bisimulation product of two trees, and also that the signature of $(\mathcal{H}, V)$ is $\langle \text{PROP} \cup \{known\}, \text{REL}\rangle$. Therefore, we can define the $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-model $(\mathcal{H}, V', S)$ over $\langle \text{PROP}, \text{REL}\rangle$ where $V' = V$ for all $p \in \text{PROP}$ and $w \in V(known)$ iff $w \in S$. It is easy to see that the equivalent $\mathcal{ML}$-model for $(\mathcal{H}, V', S)$ is $(\mathcal{H}, V)$. So now we need some claim that guarantees us that we can build two relations $\sim_f$ and $\sim_g$ from the graphs of f and g respectively, such that $\sim_f$ is an $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-bisimulation between $(\mathcal{H}, V', S)$ and $\mathcal{M}^+_T$ and $\sim_g$ is an $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-bisimulation between $(\mathcal{H}, V', S)$ and $\mathcal{N}^+_T$. We will not give the proof of this claim here (refer to [7] for more details). Assuming that we can actually build those relations, it follows that $(\mathcal{H}, V', S), \langle w, v\rangle \models \varphi \wedge \neg\psi$. This contradicts our initial assumption that $\mathcal{C} \models \varphi \to \psi$.

For the $\mathcal{ML}^m(\langle\langle r \rangle\rangle, Ⓕ)$ case. Let Tr be the equivalence preserving translation defined in [5] that takes $\mathcal{ML}^m(\langle\langle r \rangle\rangle, Ⓕ)$-formulas to $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-formulas. Observe that Tr preserves propositional symbols and known, that is, given $\varphi \in \mathcal{ML}^m(\langle\langle r \rangle\rangle, Ⓕ)$, ⓚ occurs in φ if and only if ⓚ occurs in $\mathsf{Tr}(\varphi)$ and $\mathsf{props}(\varphi) = \mathsf{props}(\mathsf{Tr}(\varphi))$.

Let φ and ψ be two $\mathcal{ML}^m(\langle\langle r \rangle\rangle, Ⓕ)$-formulas such that $\varphi \to \psi$ is valid. Using Tr, we know that $\mathsf{Tr}(\varphi) \to \mathsf{Tr}(\psi)$ is a valid $\mathcal{ML}^m(\langle\langle r \rangle\rangle)$-formula. By Theorem 12, we know that there is an interpolant χ for $\mathsf{Tr}(\varphi)$ and $\mathsf{Tr}(\psi)$ in the common language. Since Tr preserves equivalence, χ is also an interpolant for φ and ψ. Furthermore, given that Tr preserves propositional symbols and known, χ is in the common language of φ and ψ. □